TTP00-01

# Calculating three-loop diagrams in Heavy Quark Effective Theory with integration-by-parts recurrence relations*

A. G. Grozin
Institut für Theoretische Teilchenphysik, Universität Karlsruhe
grozin@particle.uni-karlsruhe.de

**Abstract**

An algorithm for calculation of three-loop propagator diagrams in HQET, based on integration-by-parts recurrence relations, is constructed and implemented as a REDUCE package Grinder, and in Axiom.

## 1 Introduction

Perturbative quantum field theory is progressing fast. New high-precision experiments require calculation of higher radiative corrections — multiloop Feynman diagrams. Recently, some calculations have been done which would seem impossible only a few years ago. This is due to the high degree of automation of the process of generation, analyses and calculation of Feynman diagrams, which is achieved via extensive use of computer algebra (see [1] for review and references). All such calculations are performed in the framework of dimensional regularization [2], i. e. diagrams are calculated as analytical functions of the space-time dimension $d = 4 - 2\epsilon$.

The integration-by-parts method [3] was invented for calculation of three-loop massless propagator diagrams. It is the most systematic method of those currently used, and the most appropriate for computer-algebra implementation. It was first implemented as a SCHOONSCHIP [4] package MINCER [5], and later re-implemented [6] in FORM [7] (in fact, FORM was created mainly to run MINCER). Since then, MINCER has been the engine behind most of

*Talk at VII Rhine Workshop on Computer Algebra, Bregenz, Austria, March 2000; to be published in the Proceedings.

spectacular successes of perturbative field theory. Some of these calculations, with gigabyte-size intermediate expressions, are among the largest computer-algebra calculations ever undertaken.

Integration-by-parts was used for other classes of problems, too. Many interesting physical results have been obtained with packages for calculating two-loop on-shell massive diagrams and three-loop vacuum diagrams with a single mass (see, e. g., [8, 9, 10]). First three-loop on-shell calculations have been done recently [11, 12].

Several years ago, an interesting new approach to heavy-quark problems in Quantum Chromodynamics has been formulated — Heavy Quark Effective Theory (HQET), see, e. g., [13, 14] for review and references. In collaboration with David Broadhurst, I applied the integration-by-parts method for calculating two-loop propagator diagrams in HQET [15]. Since then, the algorithm suggested was used in a large number of physics applications. A short review of the integration-by-parts method as applied in heavy quark physics is presented in [16].

In the present work, I apply this method for calculating three-loop propagator diagrams in HQET. Three-loop anomalous dimensions and spectral densities in HQET are necessary for a number of physics applications, such as improved extraction of the $B$ meson decay constant $f_B$ from lattice simulations and from QCD sum rules. HQET Lagrangian does not involve mass in the leading order, and, therefore, the problem is quite similar to the massless one. My aim is to produce a reliable package for three-loop HQET calculations, which could play the same role as MINCER in massless theories. I call it Grinder. Some complication comes from the fact that there are two kinds of lines now — massless propagators and infinitely-heavy ones, and hence the number of diagram topologies is substantially larger.

In Sec. 2, we consider three-loop HQET propagator diagrams with one- or two-loop massless or HQET propagator subdiagrams. To this end, we first recall well-known results for massless [3] and HQET [15] one- and two-loop diagrams. After that, diagrams with two-loop insertions, or with two one-loop insertions, are easily calculated. Two-loop HQET diagrams with a single one-loop insertion are dealt with in a manner similar to the plain two-loop diagrams. In Sec. 3, we consider proper three-loop HQET propagator diagrams. Some details of implementation and testing are presented in Sec. 4.

# 2 Diagrams with lower-loop propagator insertions

## 2.1 Two-loop massless propagator diagrams

The one-loop massless propagator integral (Fig. 1) can be easily calculated by using the Feynman parameterization or Fourier transform to the coordinate space and back:

$$\begin{aligned}
&\int \frac{d^d k}{D_1^{n_1} D_2^{n_2}} = i\pi^{d/2} (-p^2)^{d/2-n_1-n_2} G(n_1, n_2)\,, \\
&D_1 = -k^2\,, \quad D_2 = -(k+p)^2\,, \\
&G(n_1, n_2) = \frac{\Gamma(n_1+n_2-d/2)\Gamma(d/2-n_1)\Gamma(d/2-n_2)}{\Gamma(n_1)\Gamma(n_2)\Gamma(d-n_1-n_2)}\,.
\end{aligned} \tag{1}$$

The integral with a numerator can be written as a finite sum [3]

$$\begin{aligned}
&\int \frac{P_n(k) d^d k}{D_1^{n_1} D_2^{n_2}} = i\pi^{d/2} (-p^2)^{d/2-n_1-n_2} \sum_m G(n_1, n_2; n, m) \frac{(-p^2)^m}{m!} \\
&\times \left( -\frac{1}{4} \frac{\partial}{\partial k_\mu} \frac{\partial}{\partial k^\mu} \right)^m P_n(k) \bigg|_{k \to p}\,, \\
&G(n_1, n_2; n, m) = \\
&\quad \frac{\Gamma(n_1+n_2-m-d/2)\Gamma(d/2-n_1+n-m)\Gamma(d/2-n_2+m)}{\Gamma(n_1)\Gamma(n_2)\Gamma(d-n_1-n_2+n)}\,,
\end{aligned} \tag{2}$$

where $P_n(k)$ is an arbitrary homogeneous polynomial: $P_n(\lambda k) = \lambda^n P_n(k)$.

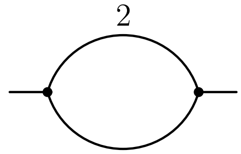



Figure 1: One-loop massless propagator diagram

We write the two-loop propagator integral (Fig. 2a) as

$$\begin{aligned}
&\int \frac{d^d k_1 d^d k_2}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5}} = -\pi^d (-p^2)^{d-\sum n_i} G(n_1, n_2, n_3, n_4, n_5)\,, \\
&D_1 = -k_1^2\,, \quad D_2 = -k_2^2\,, \quad D_3 = -(k_1+p)^2\,, \quad D_4 = -(k_2+p)^2\,, \\
&D_5 = -(k_1-k_2)^2\,.
\end{aligned} \tag{3}$$

It is symmetric with respect to $1 \leftrightarrow 2$, $3 \leftrightarrow 4$, and also $1 \leftrightarrow 3$, $2 \leftrightarrow 4$. If one of the indices is zero, it can be easily calculated using (1) (Fig. 2b, c)

$$G(n_1, n_2, n_3, n_4, 0) = G(n_1, n_3) G(n_2, n_4) \,, \tag{4}$$

$$G(0, n_2, n_3, n_4, n_5) = G(n_3, n_5) G(n_2, n_4 + n_3 + n_5 - d/2) \tag{5}$$

(and symmetric relations).

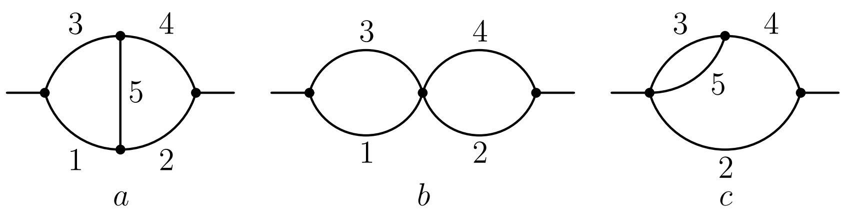



Figure 2: Two-loop massless propagator diagram

Applying the operators $\partial_1 \cdot (k_1 - k_2)$ and $\partial_1 \cdot k_1$ (where $\partial_i = \frac{\partial}{\partial k_i}$) to the integrand of (3), we obtain the recurrence relations for $G(n_1, n_2, n_3, n_4, n_5)$ (known as triangle relations [3])

$$\left[d - n_1 - n_3 - 2n_5 + n_1 \mathbf{1}^+ (\mathbf{2}^- - \mathbf{5}^-) + n_3 \mathbf{3}^+ (\mathbf{4}^- - \mathbf{5}^-)\right] G = 0 \,, \tag{6}$$

$$\left[d - n_3 - n_5 - 2n_1 + n_3 \mathbf{3}^+ (1 - \mathbf{1}^-) + n_5 \mathbf{5}^+ (\mathbf{2}^- - \mathbf{1}^-)\right] G = 0 \,, \tag{7}$$

where, for example,

$$\mathbf{1}^\pm G(n_1, n_2, n_3, n_4, n_5) = G(n_1 \pm 1, n_2, n_3, n_4, n_5) \,. \tag{8}$$

Of course, more relations are obtained by symmetry. Another interesting relation is derived by applying the operator $\frac{\partial}{\partial p} \cdot (k_2 + p)$. Substituting the general form of the relevant vector integral, we arrive at

$$\begin{aligned} &\Big[\tfrac{1}{2} d + n_4 - n_1 - n_2 - n_5 + \left(\tfrac{3}{2} d - n_1 - n_2 - n_3 - n_4 - n_5\right) (\mathbf{4}^- - \mathbf{2}^-) \\ &\quad + n_3 \mathbf{3}^+ (\mathbf{4}^- - \mathbf{5}^-)\Big] G = 0 \,. \end{aligned} \tag{9}$$

This formula was derived long ago by S. A. Larin in his M. Sc. thesis [17] (again, similar relations follow by symmetries).

If indices of two adjacent lines are non-positive integers, the integral contains a no-scale vacuum subdiagram and hence vanishes. The cases with zero indices are given by (4), (5). When $n_5 < 0$ and $n_3 \neq 1$, $n_5$ can be raised by (9); the cases $n_1 \neq 1$, $n_2 \neq 1$, $n_4 \neq 1$ are symmetric. The case $n_5 < 0$, $n_1 = n_1 = n_3 = n_4 = 1$ is handled by

$$\begin{aligned} &\left[(d - 2n_5 - 4) \mathbf{5}^+ + 2(d - n_5 - 3)\right] G(1, 1, 1, 1, n_5) \\ &\quad = 2 \mathbf{1}^+ (\mathbf{3}^- - \mathbf{2}^- \mathbf{5}^+) G(1, 1, 1, 1, n_5) \,, \end{aligned} \tag{10}$$

which follows from (6), (7) at $n_1 = n_2 = n_3 = n_4 = 1$ (note that the terms in the right-hand side of (10) are trivial for any $n_5$; for $n_5 < 0$, they vanish). When $n_2 < 0$, it can be raised by (9); the cases $n_1 < 0$, $n_3 < 0$, $n_4 < 0$ are symmetric.

We are left with the most important situation when all the indices are positive. Applying (6), we reduce $n_2$, $n_4$, $n_5$ until one of them vanishes. Then (4), (5) apply. If $\max(n_1, n_3) < \max(n_2, n_4)$, it is more efficient to lower $n_1$, $n_3$, $n_5$.

All one-loop integrals (Fig. 1) with integer $n_{1,2}$ are proportional to $G_1 = G(1,1)$, the coefficient being a rational function of $d$. All two-loop integrals with integer indices reduce to $G_1^2$ (Fig. 3a) and $G_2 = G(0,1,1,0,1)$ (Fig. 3b), with rational coefficients. Here

$$G_n = \frac{1}{\left(n+1-n\frac{d}{2}\right)_n \left((n+1)\frac{d}{2}-2n-1\right)_n} \frac{\Gamma(1+n\epsilon)\Gamma^{n+1}(1-\epsilon)}{\Gamma(1-(n+1)\epsilon)}\,, \qquad (11)$$

where $(x)_n = \Gamma(x+n)/\Gamma(x)$ is the Pochhammer symbol.

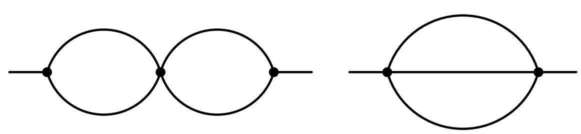



Figure 3: Basis two-loop massless propagator integrals

## 2.2 Two-loop HQET propagator diagrams

The one-loop HQET propagator integral (Fig. 4) can be easily calculated by using the modified Feynman parameterization (see, e. g., [15, 16]), or Fourier transform to the coordinate space and back:

$$\begin{aligned}
&\int \frac{d^d k}{D_1^{n_1} D_2^{n_2}} = i\pi^{d/2} (-2\omega)^{d-2n_2} I(n_1,n_2)\,,\\
&D_1 = (k+p)\cdot v/\omega\,, \quad D_2 = -k^2\,,\\
&I(n_1,n_2) = \frac{\Gamma(n_1+2n_2-d)\Gamma(d/2-n_2)}{\Gamma(n_1)\Gamma(n_2)}
\end{aligned} \qquad (12)$$

(here $v$ is 4-velocity of the heavy quark, $v^2 = 1$, and $\omega = p\cdot v$ is the residual energy). Similarly to (2), we obtain

$$\int \frac{P_n(k) d^d k}{D_1^{n_1} D_2^{n_2}} = i\pi^{d/2} (-2\omega)^{d-2n_2} \sum_m I(n_1,n_2;n,m) \frac{(-2\omega)^{2m}}{m!}$$

$$\times \left(-\frac{1}{4}\frac{\partial}{\partial k_\mu}\frac{\partial}{\partial k^\mu}\right)^m P_n(k)\Bigg|_{k\to 2\omega v},$$

$$I(n_1,n_2;n,m) = \frac{\Gamma(n_1+2n_2-n-d)\Gamma(d/2-n_2+n-m)}{\Gamma(n_1)\Gamma(n_2)}\,. \quad (13)$$

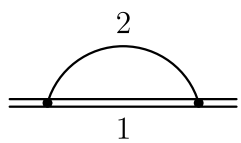



Figure 4: One-loop HQET propagator diagram

There are two topologies of two-loop propagator HQET diagrams (Fig. 5a, b). We write the first of them as

$$\int \frac{d^d k_1 d^d k_2}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5}} = -\pi^d (-2\omega)^{2(d-n_3-n_4-n_5)} I(n_1,n_2,n_3,n_4,n_5)\,,$$
$$D_1 = (k_1+p)\cdot v/\omega\,,\quad D_2 = (k_2+p)\cdot v/\omega\,,$$
$$D_3 = -k_1^2\,,\quad D_4 = -k_2^2\,,\quad D_5 = -(k_1-k_2)^2\,. \quad (14)$$

It is symmetric with respect to $1 \leftrightarrow 3$, $2 \leftrightarrow 4$. If one of the indices is zero, it can be easily calculated using (12), (1) (Fig. 5c, d, e)

$$I(n_1,n_2,n_3,n_4,0) = I(n_1,n_3)I(n_2,n_4)\,, \quad (15)$$
$$I(0,n_2,n_3,n_4,n_5) = G(n_3,n_5)I(n_2,n_4+n_3+n_5-d/2)\,, \quad (16)$$
$$I(n_1,n_2,0,n_4,n_5) = I(n_1,n_5)I(n_2+n_1+2n_5-d,n_4) \quad (17)$$

(and symmetric relations).

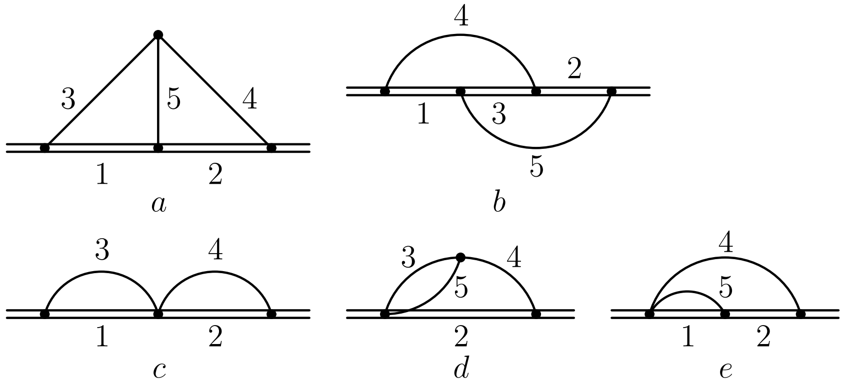



Figure 5: Two-loop HQET propagator diagram

Applying the operators $\partial_1 \cdot (k_1 - k_2)$, $\partial_1 \cdot k_1$ and $\partial_1 \cdot v$ to the integrand of (14), we obtain [15]

$$\left[d - n_1 - n_3 - 2n_5 + n_1 \mathbf{1}^+ \mathbf{2}^- + n_3 \mathbf{3}^+ (\mathbf{4}^- - \mathbf{5}^-)\right] I = 0\,, \tag{18}$$

$$\left[d - n_1 - n_5 - 2n_3 + n_1 \mathbf{1}^+ + n_5 \mathbf{5}^+ (\mathbf{4}^- - \mathbf{3}^-)\right] I = 0\,, \tag{19}$$

$$\left[-2n_1 \mathbf{1}^+ + n_3 \mathbf{3}^+ (\mathbf{1}^- - 1) + n_5 \mathbf{5}^+ (\mathbf{1}^- - \mathbf{2}^-)\right] I = 0\,. \tag{20}$$

Of course, more relations are obtained by symmetry. Applying $\omega \frac{d}{d\omega}$ and using homogeneity in $\omega$, we obtain

$$\left[2(d - n_3 - n_4 - n_5) - n_1 - n_2 + n_1 \mathbf{1}^+ + n_2 \mathbf{2}^+\right] I = 0\,, \tag{21}$$

which is nothing but the sum of (19) and its mirror-symmetric. Subtracting the $\mathbf{2}^-$ shifted version of (21) from (18), we obtain the most useful relation [15]

$$\begin{aligned}&\Bigl[d - n_1 - n_2 - n_3 - 2n_5 + 1 \\ &\quad - (2(d - n_3 - n_4 - n_5) - n_1 - n_2 + 1)\mathbf{2}^- + n_3 \mathbf{3}^+ (\mathbf{4}^- - \mathbf{5}^-)\Bigr] I = 0\,,\end{aligned} \tag{22}$$

which lowers $n_2$, $n_4$, $n_5$, and does not raise heavy-quark indices.

If indices of two adjacent lines are non-positive integers, the integral contains a no-scale vacuum subdiagram and hence vanishes. The cases with zero indices are given by (15), (16), (17). When $n_2 < 0$, it can be raised by (22); the case $n_1 < 0$ is symmetric. Similarly, if $n_3 < 0$, it can be raised by (22); the case $n_4 < 0$ is symmetric. When $n_5 < 0$ and $n_3 \neq 1$, $n_5$ can be raised by (22) (the case $n_4 \neq 1$ is symmetric); when $n_5 < 0$ and $n_1 \neq 1$, $n_5$ can be raised by (19) (the case $n_2 \neq 1$ is symmetric). The case $n_5 < 0$, $n_1 = n_1 = n_3 = n_4 = 1$ is handled by

$$\begin{aligned}&\Bigl[(d - 2n_5 - 4)\mathbf{5}^+ - 2(d - n_5 - 3) \\ &\quad - (2d - 2n_5 - 7)\mathbf{1}^- \mathbf{5}^+ + \mathbf{3}^- \mathbf{4}^+ \mathbf{5}^- - \mathbf{1}^- \mathbf{3}^+\Bigr] I(1,1,1,1,n_5) = 0\,,\end{aligned} \tag{23}$$

which follows from (18), (19), (20) at $n_1 = n_2 = n_3 = n_4 = 1$ (note that the terms on the second line of (23) are trivial for any $n_5$; for $n_5 < 0$, they vanish).

We are left with the most important situation when all the indices are positive. Applying (22), we reduce $n_2$, $n_4$, $n_5$ until one of them vanishes. Then (15), (16), (17) apply. If $\max(n_1, n_3) < \max(n_2, n_4)$, it is more efficient to lower $n_1$, $n_3$, $n_5$.

In the second topology (Fig. 5b), three heavy-quark denominators depend on only two variables $k_{1,2} \cdot v$, hence they are linearly dependent. Therefore, there is one scalar product which cannot be expressed via the denominators. Let's define the integral

$$
\begin{aligned}
&\int \frac{N^{n_0} d^d k_1 d^d k_2}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5}} = -\pi^d (-2\omega)^{2(d-n_3-n_4-n_5)} J(n_1,n_2,n_3,n_4,n_5;n_0)\,, \\
&D_1 = (k_1+p)\cdot v/\omega\,, \quad D_2 = (k_2+p)\cdot v/\omega\,, \quad D_3 = (k_1+k_2+p)\cdot v/\omega\,, \\
&D_4 = -k_1^2\,, \quad D_5 = -k_2^2\,, \quad N = 2k_1\cdot k_2
\end{aligned}
\tag{24}
$$

(it is symmetric with respect to $1 \leftrightarrow 2$, $4 \leftrightarrow 5$). Noting that $D_1+D_2-D_3=1$, we immediately have [15]

$$
(1-\mathbf{1}^- - \mathbf{2}^- + \mathbf{3}^-)J = 0\,. \tag{25}
$$

Applying $\partial_1 \cdot k_1$, $\partial_1 \cdot k_2$ and $\partial_1 \cdot v$ to the integrand, we have

$$
\left[d + n_0 - n_1 - n_3 - 2n_4 + n_1\mathbf{1}^+ + n_3\mathbf{3}^+\mathbf{2}^-\right] J = 0\,, \tag{26}
$$

$$
\left[n_1 - n_3 - n_1\mathbf{1}^+\mathbf{3}^- + n_3\mathbf{3}^+\mathbf{1}^- + n_4\mathbf{4}^+\mathbf{0}^+ - 2n_0\mathbf{0}^-\mathbf{5}^-\right] J = 0\,, \tag{27}
$$

$$
\left[-2n_1\mathbf{1}^+ - 2n_3\mathbf{3}^+ + n_4\mathbf{4}^+(\mathbf{1}^- - 1) + n_0\mathbf{0}^-(\mathbf{2}^- - 1)\right] J = 0\,. \tag{28}
$$

Homogeneity in $\omega$ gives $[2(d+n_0-n_4-n_5) - n_1 - n_2 - n_3 + n_1\mathbf{1}^+ + n_2\mathbf{2}^+ + n_3\mathbf{3}^+]J = 0$, which is nothing but the sum of (26) and its mirror-symmetric. The boundary values of the integral (Fig. 5c, e) are

$$
J(n_1,n_2,0,n_4,n_5;n_0) = (\mathbf{5}^- - \mathbf{3}^- - \mathbf{4}^-)^{n_0} I(n_1,n_2,n_4,n_5,0)\,, \tag{29}
$$

$$
J(0,n_2,n_3,n_4,n_5;n_0) = (\mathbf{4}^- - \mathbf{3}^- - \mathbf{5}^-)^{n_0} I(n_3,n_2,0,n_5,n_4) \tag{30}
$$

and the symmetric relation for $n_2 = 0$. If $n_0 = 0$,

$$
J(n_1,n_2,0,n_4,n_5) = I(n_1,n_4) I(n_2,n_5)\,, \tag{31}
$$

$$
J(0,n_2,n_3,n_4,n_5) = I(n_3,n_4) I(n_2+n_3+2n_4-d,n_5)\,. \tag{32}
$$

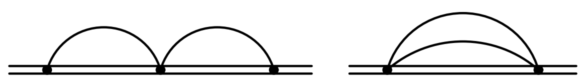
If $n_4 \le 0$, or $n_5 \le 0$, or two adjacent heavy-quark indices are non-positive, the integral vanishes. If any of $n_1$, $n_2$, $n_3$ is negative, it can be raised by (25). If all of them are positive, we use (25) to lower $n_1$, $n_2$ or $n_3$, until one of these indices vanish.

Instead of using (29), (30) when $n_0 > 0$, we could proceed in another way. If $n_4 > 1$, we lower both $n_0$ and $n_4$ using (27) (the case $n_5 > 1$ is symmetric). We are left with $J(n_1,n_2,0,1,1;n_0)$ and $J(0,n_2,n_3,1,1;n_0)$. In the first case, if $n_1 > 1$, we lower it using (28) (the case $n_2 > 1$ is symmetric). In the second case, if $n_3 > 1$, we lower it using (28); if $n_2 > 1$, we lower it using the difference of (28) and its mirror-symmetric. The two remaining cases (Fig. 5c, e) are easily evaluated using (13):

$$J(1,1,0,1,1;n_0) = (-1)^{n_0} I(1,1;n_0,0) I(1,1)\,,$$
$$J(0,1,1,1,1;n_0) = (-1)^{n_0} I(1,1;n_0,0) I(4-d,1)\,.$$

All one-loop integrals (Fig. 4) with integer $n_{1,2}$ are proportional to $I_1 = I(1,1)$, the coefficient being a rational function of $d$. All two-loop integrals with integer indices reduce to $I_1^2$ (Fig. 6a) and $I_2 = I(0,1,1,0,1)$ (Fig. 6b), with rational coefficients. Here

$$I_n = \frac{1}{(1-n(d-2))_{2n}} \Gamma(1+2n\epsilon)\Gamma^n(1-\epsilon)\,. \tag{33}$$



*a* *b*

Figure 6: Basis two-loop HQET propagator integrals

## 2.3 Three-loop HQET diagrams with lower-loop insertions

Diagrams with a two-loop propagator insertion, or with two one-loop insertions, Fig. 7, are trivially calculated by multiplying the relevant insertion[s] by the one-loop HQET integral with non-integer indices, whose values are obvious by dimensionality.

Next we consider the diagrams obtained from the two-loop HQET diagram of Fig. 5a by adding a single one-loop propagator insertion (Fig. 8). They are equal to the product of the corresponding one-loop integral and the integral of Fig. 5a with one non-integer index. All relations derived for the diagram of Fig. 5a are valid; however, the non-integer index changes the strategy of their application.

Let's consider the case of Fig. 8a with a non-integer $n_3$. When $n_5 < 0$, it can always be raised by (22). When $n_2 < 0$, it can be raised by (22), too; when $n_1 < 0$, it can be raised by the mirror-symmetric relation. When $n_4 < 0$, it can always be raised by (22), too. Finally, when all indices $n_1$, $n_2$, $n_4$, $n_5$ are positive, we use (22) to lower $n_2$, $n_4$ or $n_5$, until (15), (16), (17) are reached.

In the case of Fig. 8b, $n_1$ is non-integer. When $n_5 < 0$ and $n_3 \neq 1$, we can raise $n_5$ by (22) (the case $n_4 \neq 1$ is symmetric); when $n_5 < 0$ and

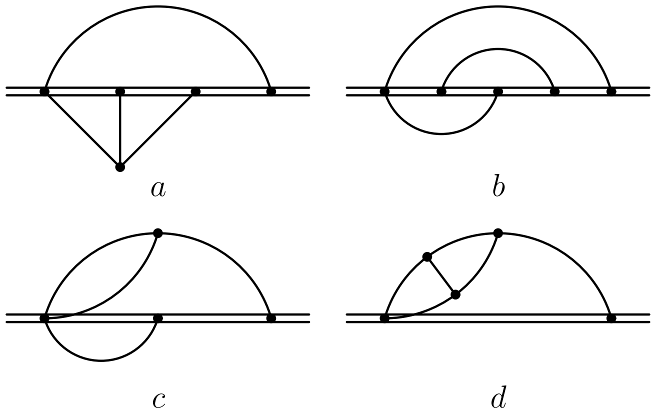


Figure 7: Three-loop HQET propagator diagrams with a two-loop propagator insertion, or with two one-loop insertions

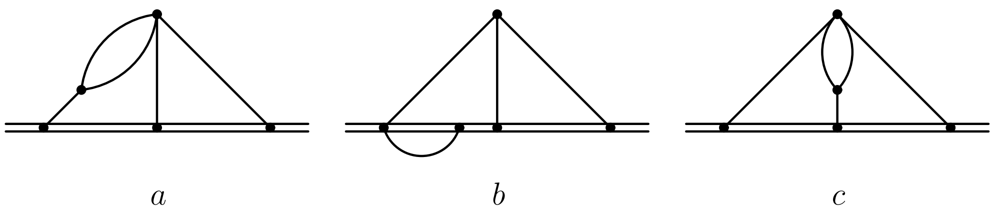


Figure 8: The diagram of Fig. 5a with a single one-loop insertion

$n_2 \neq 1$, we can raise $n_5$ by the relation symmetric to (19); when $n_5 < 0$ and $n_2 = n_3 = n_4 = 1$, we can use the relation

$$\left[2(d - n_5 - 3) - \mathbf{4}^+ - n_5 \mathbf{5}^+ \mathbf{1}^-\right] I(n_1, 1, 1, 1, n_5) = 0\,, \tag{34}$$

which follows from (20) and (19), and then the term with $\mathbf{4}^+$ can be treated as above. When $n_2 < 0$, it can be raised by (22). When $n_3 < 0$ and $n_4 \neq 1$, we can raise $n_3$ by the relation symmetric to (22); when $n_3 < 0$ and $n_5 \neq 1$, we can raise $n_3$ by (19); when $n_3 < 0$ and $n_2 \neq 1$, we lower $n_2$ by (21); finally, when $n_3 < 0$ and $n_2 = n_4 = n_5 = 1$, we use (22) to raise $n_3$ and get a trivial additional term. When $n_4 < 0$ and $n_3 \neq 1$, we can raise $n_4$ by (22); when $n_4 < 0$ and $n_5 \neq 1$, we can raise $n_4$ by (19); when $n_4 < 0$ and $n_2 \neq 1$, we lower $n_2$ by (21); finally, when $n_4 < 0$ and $n_2 = n_3 = n_5 = 1$, we use (20) to raise $n_3$ and get trivial additional terms, and proceed as above. Finally, when all indices $n_2$, $n_3$, $n_4$, $n_5$ are positive, we use (22) to lower $n_2$, $n_4$ or $n_5$, until (15), (16), (17) are reached.

In the case of Fig. 8c, $n_5$ is non-integer. When $n_2 < 0$, it can be raised by (22); the case $n_1 < 0$ is symmetric. When $n_3 < 0$, it can be raised by (19); the case $n_4 < 0$ is symmetric. When $n_1 > 1$, it can be lowered by (19); the case $n_2 > 1$ is symmetric. When $n_3 > 1$, it can be lowered by (20), with a trivial additional term having $n_1 = 0$; the case $n_4 > 1$ is symmetric. We are left with $n_1 = n_2 = n_3 = n_4 = 1$; the relation (23) can be used to lower or raise $n_5$, with trivial additional terms. An integral with some specific value of $n_5$ (of the form integer plus $\epsilon$) has to be considered as a new basis element. This integral has been calculated, exactly in $d$ dimensions, in [18] in terms of ${}_3F_2$ hypergeometric functions, using the Hegenbauer polynomial technique in the coordinate space [19].

Finally, we consider the diagrams obtained from the two-loop HQET diagram of Fig. 5b by adding a single one-loop propagator insertion (Fig. 9). The diagram of Fig. 9a, with a non-integer $n_4$ is calculated exactly as the two-loop one.

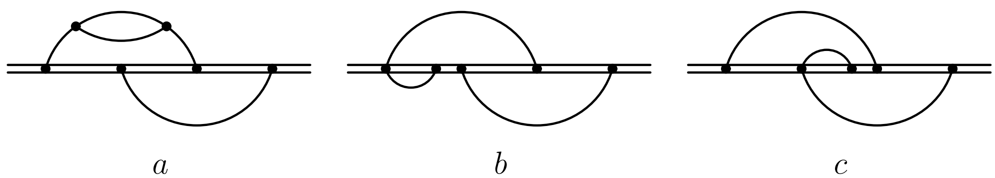



Figure 9: The diagram of Fig. 5b with a single one-loop insertion

In the case of Fig. 9b, $n_1$ is non-integer. When $n_3 < 0$ or $n_2 < 0$, they can be raised by (25). In the case of positive indices, we use

$$\left[d - n_3 - 2n_4 + n_1 \mathbf{1}^+(\mathbf{2}^- - \mathbf{3}^-) + n_3 \mathbf{3}^+ \mathbf{2}^-\right] J = 0 \qquad (35)$$

(which follows from (26), (25)). It either lowers $n_2 + n_3$, or, at a fixed $n_2 + n_3$, lowers $n_2$. Therefore, sooner or later, we reach (29), (30).

In the case of Fig. 9c, $n_3$ is non-integer. If $n_1 < 0$ or $n_2 < 0$, they can be raised by (25). When $n_4 > 1$, we can lower it or $n_1$ by (27); the case $n_5 > 1$ is symmetric. When $n_1 > 1$, it can be lowered by (26); the case $n_2 > 1$ is symmetric. We are left with $n_1 = n_2 = n_4 = n_5 = 1$; the relation (25) can be used to lower or raise $n_3$, with trivial additional terms. An integral with some specific value of $n_3$ (of the form integer plus $2\epsilon$) has to be considered as a new basis element.

It is not difficult to calculate $J(1, 1, n_3, n_4, n_5)$ for arbitrary $n_{3,4,5}$ (not necessarily integer) in the coordinate space:

$$J(1, 1, n, n_1, n_2) = \frac{\Gamma(n - 2(d - n_1 - n_2 - 1))\Gamma(d/2 - n_1)\Gamma(d/2 - n_2)}{\Gamma(n)\Gamma(n_1)\Gamma(n_2)} J\,,$$
$$J = t^{2(d-n_1-n_2)-n-1} \int\limits_{0<t_1<t_2<t} dt_1\, dt_2\, t_2^{2n_1-d} (t - t_1)^{2n_2-d} (t_2 - t_1)^{n-1}$$

$$= \frac{1}{n(2n_1+n+1-d)}\, {}_3F_2\left( \begin{array}{c} 1,\ d-2n_2,\ 2n_1+n+1-d \\ n+1,\ 2n_1+n+2-d \end{array} \middle|\, 1 \right). \qquad (36)$$

All diagrams considered in this Section are particular cases of the generic three-loop topologies, which will be discussed in Sec. 3, when some lines are shrunk (i. e., some indices vanish). Therefore, we don't consider diagrams with numerators here: numerators should be dealt with in the context of generic topologies, and the formulae of this Section are used only as boundary values for the corresponding recurrence relations, after elimination of numerators.

All three-loop HQET propagator integrals with lower-loop propagator insertions are linear combinations of 7 basis integrals (Fig. 10a–g), coefficients being rational functions of $d$. The basis integrals of Fig. 10a ($I_1^3$), Fig. 10b ($I_1 I_2$), Fig. 10c ($I_3$), Fig. 10d ($I_3 I_1^2/I_2$), and Fig. 10e ($I_3 G_1^2/G_2$) are known exactly in $d$ dimensions in terms of $\Gamma$ functions. Those of Fig. 10f, g contain hypergeometric ${}_3F_2$ functions of the unit argument. Several terms of their expansion in $\epsilon$ can be, probably, obtained using the methods which were recently developed in [20]. As we shall see in Sec. 3, there is only one additional basis integral, Fig. 10h.

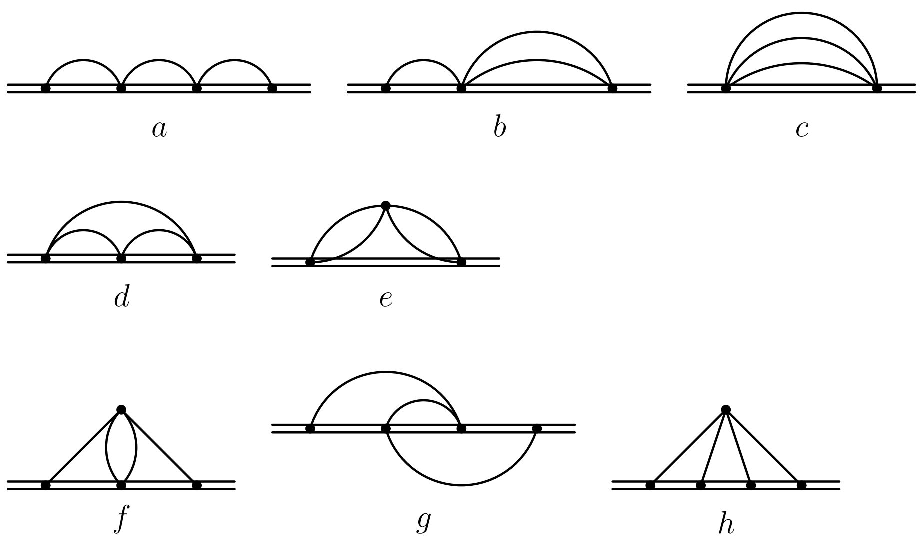



Figure 10: Basis tree-loop HQET propagator integrals

# 3 Proper three-loop HQET propagator diagrams

In the massless case [3], there are only 3 topologies of proper three-loop propagator diagrams: Mercedez, Ladder, and Non-planar (plus their reduced forms obtained by shrinking some lines). Now we have 10 topologies instead (Fig. 11). Each of them has 8 propagators. In the diagrams with 4 heavy-quark lines (Fig. 11e–g), there is one linear dependence between their denominators; with 5 heavy-quark lines (Fig. 11h–j) — 2 dependences. There are 9 independent scalar products of 3 loop momenta $k_{1,2,3}$ and the 4-velocity $v$. Therefore, in the diagrams with two or three heavy-quark lines (Fig. 11a–d), there is one scalar product in the numerator which cannot be cancelled against the denominators; with 4 heavy-quark lines (Fig. 11e–g) — two scalar products; with 5 heavy-quark lines (Fig. 11h–j) — three scalar products.

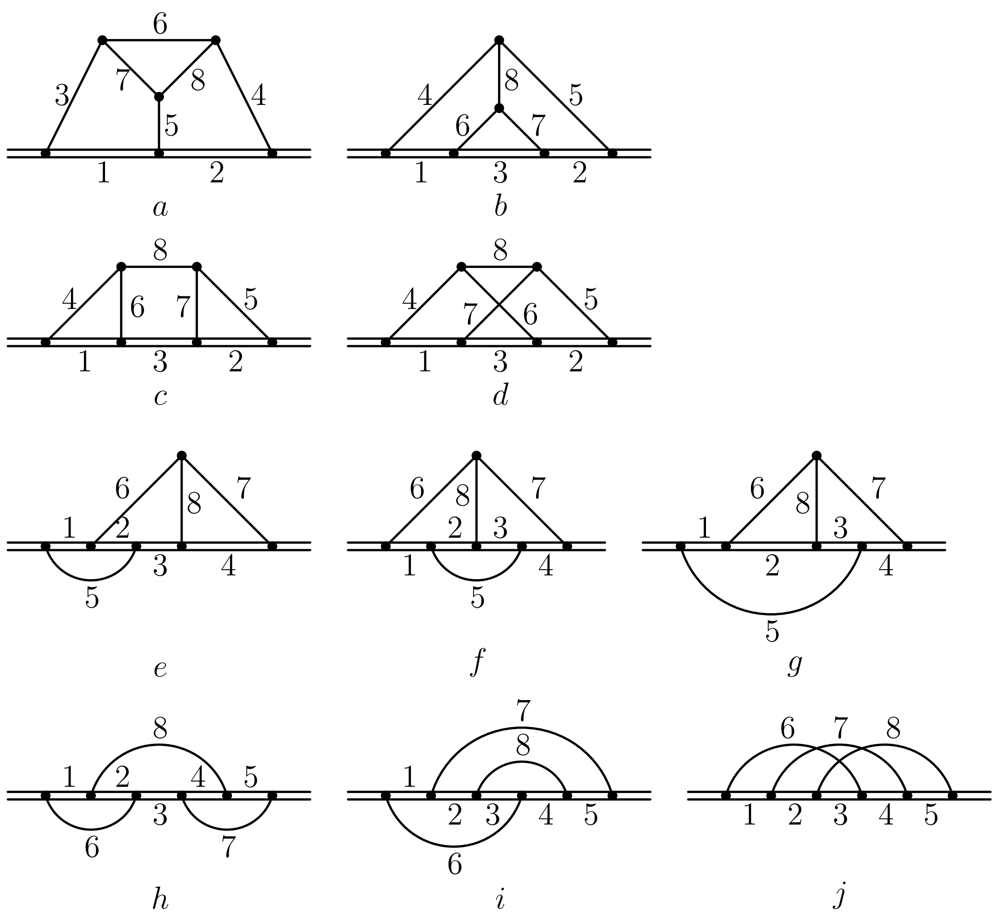



Figure 11: Topologies of proper three-loop HQET propagator diagrams: Mercedez (a, b, f, g, i), Ladder (c, e, h), Non-planar (d, j)

When calculating these diagrams using recurrence relations, some indices may vanish. This corresponds to shrinking the corresponding lines. In some cases, this results in diagrams with lower-loop propagator insertions, which were calculated in Sec. 2. The diagrams of Fig. 12 are still non-trivial.

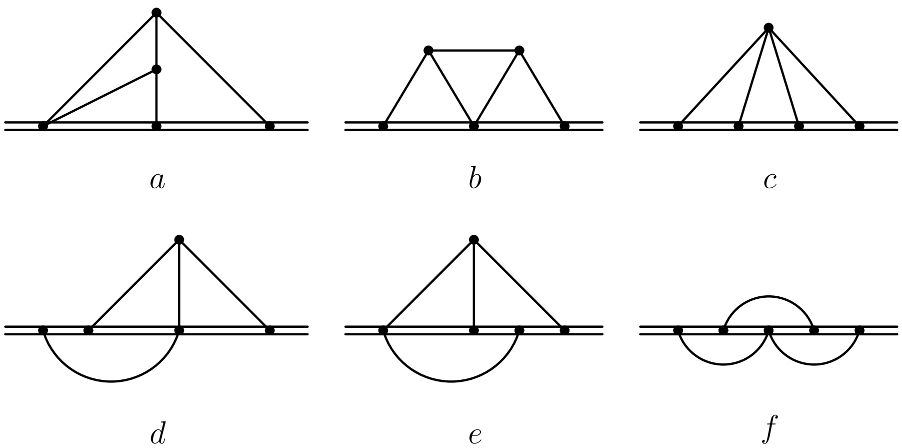



Figure 12: Topologies of non-trivial three-loop diagrams with a shrunk line

## 3.1 Diagram with two heavy-quark lines

Let's consider the diagram of Fig. 11a first. We define

$$\begin{aligned}
&\int \frac{N^{n_0} d^d k_1 d^d k_2 d^d k_3}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7} D_8^{n_8}} \\
&= -i\pi^{3d/2} (-2\omega)^{3d+2n_0-2\sum_{i=3}^{8} n_i} I_a(n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8; n_0)\,, \\
&N = k_3 \cdot v/\omega\,, \quad D_1 = (k_1 + p) \cdot v/\omega\,, \quad D_2 = (k_2 + p) \cdot v/\omega\,, \\
&D_3 = -k_1^2\,, \quad D_4 = -k_2^2\,, \quad D_5 = -(k_1 - k_2)^2\,, \\
&D_6 = -k_3^3\,, \quad D_7 = -(k_3 + k_1)^2\,, \quad D_8 = -(k_3 + k_2)^2\,.
\end{aligned} \tag{37}$$

This integral is mirror-symmetric with respect to $1 \leftrightarrow 2$, $3 \leftrightarrow 4$, $7 \leftrightarrow 8$. It vanishes when the indices of the following groups of lines are non-positive: 12, 67, 68, 78, 375, 485, 315, 425, 364, 137, 248, 157, or 258.

First we are going to get rid of the numerator. When $n_0 > 0$ and $n_7 \neq 1$, we can lower $n_0$ by

$$\left[-2n_1 \mathbf{1}^+ + n_3 \mathbf{3}^+ (\mathbf{1}^- - 1) + n_5 \mathbf{5}^+ (\mathbf{1}^- - \mathbf{2}^-) + n_7 \mathbf{7}^+ (\mathbf{1}^- - 1 + \mathbf{0}^+)\right] I_a = 0\,, \tag{38}$$

which is obtained by applying $\partial_1 \cdot v$ to the integrand of (37); the case $n_0 > 0$, $n_8 \neq 1$ is symmetric. When $n_0 > 0$ and $n_6 \neq 1$, we can lower $n_0$ by

$$\Big[2\,[3d - n_1 - n_2 - 2(n_3 + n_4 + n_5 + n_6 + n_7 + n_8 - n_0)] \\ + n_3\mathbf{3}^+(\mathbf{1}^- - 1) + n_4\mathbf{4}^+(\mathbf{2}^- - 1) - n_6\mathbf{6}^+\mathbf{0}^+ - 2n_0\mathbf{0}^-\Big] I_a = 0\,, \quad (39)$$

which is the $(\partial_1 + \partial_2 - \partial_3)\cdot v$ relation simplified using the homogeneity relation

$$\Big[3d - n_1 - n_2 - 2(n_3 + n_4 + n_5 + n_6 + n_7 + n_8 - n_0) + n_1\mathbf{1}^+ + n_2\mathbf{2}^+\Big]\, I_a = 0\,. \quad (40)$$

When $n_0 > 0$ and $n_1 \neq 1$, we can lower $n_0$ by

$$\Big[d - n_1 - n_3 - n_5 - 2n_7 + n_1\mathbf{1}^+(1 - \mathbf{0}^+) \\ + n_3\mathbf{3}^+(\mathbf{6}^- - \mathbf{7}^-) + n_5\mathbf{5}^+(\mathbf{8}^- - \mathbf{7}^-)\Big] I_a = 0\,, \quad (41)$$

which is obtained by applying $\partial_1 \cdot (k_1 + k_3)$ to the integrand of (37); the case $n_0 > 0$, $n_2 \neq 1$ is symmetric. We are left with $n_0 > 0$, $n_1 = n_2 = n_6 = n_7 = n_8 = 1$; we use $\partial_3 \cdot (k_3 + k_1)$ relation

$$\Big[d + n_0 - n_6 - n_8 - 2n_7 + n_0\mathbf{0}^-(\mathbf{1}^- - 1) \\ + n_6\mathbf{6}^+(\mathbf{3}^- - \mathbf{7}^-) + n_8\mathbf{8}^+(\mathbf{5}^- - \mathbf{7}^-)\Big] I_a = 0 \quad (42)$$

to lower $n_0$ or raise $n_6$ or $n_8$, and apply the method described above again.

Now we shall discuss the integral $I_a(n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8)$ without numerator ($n_0 = 0$). Applying $\partial_3 \cdot k_3$, $\partial_1 \cdot k_1$, $\partial_1 \cdot (k_1 - k_2)$, $(\partial_1 + \partial_2 - \partial_3) \cdot k_1$ to the integrand of (37), we obtain the recurrence relations

$$\Big[d - n_7 - n_8 - 2n_6 + n_7\mathbf{7}^+(\mathbf{3}^- - \mathbf{6}^-) + n_8\mathbf{8}^+(\mathbf{4}^- - \mathbf{6}^-)\Big]\, I_a = 0\,, \quad (43)$$

$$\Big[d - n_1 - n_5 - n_7 - 2n_3 + n_1\mathbf{1}^+ \\ + n_5\mathbf{5}^+(\mathbf{4}^- - \mathbf{3}^-) + n_7\mathbf{7}^+(\mathbf{6}^- - \mathbf{3}^-)\Big] I_a = 0\,, \quad (44)$$

$$\Big[d - n_1 - n_3 - n_7 - 2n_5 + n_1\mathbf{1}^+\mathbf{2}^- \\ + n_3\mathbf{3}^+(\mathbf{4}^- - \mathbf{5}^-) + n_7\mathbf{7}^+(\mathbf{8}^- - \mathbf{5}^-)\Big] I_a = 0\,, \quad (45)$$

$$\Big[2(d - n_5 - n_7 - n_8) - n_4 - n_6 + n_2\mathbf{2}^+\mathbf{1}^- + n_4\mathbf{4}^+(\mathbf{3}^- - \mathbf{5}^-) \\ + n_6\mathbf{6}^+(\mathbf{3}^- - \mathbf{7}^-)\Big] I_a = 0\,, \quad (46)$$

where the last relation was simplified using (40).

The cases $n_1 = 0$, $n_2 = 0$, $n_6 = 0$, $n_7 = 0$, $n_8 = 0$ are trivial. When $n_1 < 0$, it can be raised by

$$\begin{aligned}&[3d - n_1 - n_2 - 2(n_3 + n_4 + n_5 + n_6 + n_7 + n_8)]\, I_a = \\ &\Big[d - n_1 - n_2 - n_4 - n_8 - 2n_5 + n_4\mathbf{4}^+(\mathbf{3}^- - \mathbf{5}^-) + n_8\mathbf{8}^+(\mathbf{7}^- - \mathbf{5}^-)\Big]\, \mathbf{1}^+ I_a\,,\end{aligned} \tag{47}$$

which is the difference of (40) and $\mathbf{1}^+$ shifted version of the relation symmetric to (45); the case $n_2 < 0$ is symmetric. When $n_6 < 0$ and $n_7 \neq 1$, we can raise $n_6$ by (44) (the case $n_8 \neq 1$ is symmetric); when $n_6 < 0$ and $n_7 = n_8 = 1$, we can raise $n_6$ or $n_8$ by (42). When $n_7 < 0$ and $n_8 \neq 1$, we can raise $n_7$ by the relation symmetric to (45); when $n_7 < 0$ and $n_6 \neq 1$, we can raise $n_7$ by (46); when $n_7 < 0$ and $n_6 = n_8 = 1$, we can raise $n_7$ or $n_6$ by the relation symmetric to (42). The case $n_8 < 0$ is symmetric. The case $n_3 = 0$ (Fig. 12a, $J_a(n_1, n_2, n_4, n_5, n_6, n_7, n_8)$) will be considered later in this Section; the case $n_4 = 0$ is symmetric. When $n_3 < 0$ and $n_6 \neq 1$, we can raise $n_3$ by (42); when $n_3 < 0$ and $n_7 \neq 1$, we can raise $n_3$ by (43); when $n_3 < 0$ and $n_6 = n_7 = 1$, we can raise $n_6$ or $n_7$ by the relation symmetric to (42). The case $n_4 < 0$ is symmetric. The case $n_5 = 0$ (Fig. 12b, $J_b(n_1, n_2, n_3, n_4, n_6, n_7, n_8)$) will be considered later in this Section. When $n_5 < 0$ and $n_8 \neq 1$, we can raise $n_5$ by (42) (the case $n_7 \neq 1$ is symmetric); when $n_5 < 0$ and $n_7 = n_8 = 1$, we can raise $n_7$ or $n_8$ by (43). When all the indices are positive, we can use (43) to kill one of the lines 3, 4, 6; or we can use (42) to kill one of the lines 3, 5, 7; or we can use the symmetric relation to kill one of the lines 4, 5, 8; or we can use (46) to kill one of the lines 1, 3, 5, 7; or we can use the symmetric relation to kill one of the lines 2, 4, 5, 8.

Now we consider $J_a(n_1, n_2, n_4, n_5, n_6, n_7, n_8) = I_a(n_1, n_2, 0, n_3, n_4, n_5, n_6, n_7, n_8)$ (Fig. 12a). This integral vanishes when the indices of the following groups of lines are non-positive: 12, 67, 68, 78, 57, 15, 17, 46, 485, 425, 248, or 258. If any index is zero, the integral becomes trivial. If $n_1 < 0$, $n_2 < 0$, $n_4 < 0$, $n_6 < 0$, $n_7 < 0$, or $n_8 < 0$, we just consider $J_a$ as $I_a$ with $n_3 = 0$ and proceed as usual; the integral reduces to trivial ones not including $J_a$. When $n_4 > 0$, we can kill one of the lines 2, 4, 8 using the relation symmetric to (46) (or, if $n_5 > 0$, we can kill one of the lines 2, 5, 8 using (45)).

Finally, we consider $J_b(n_1, n_2, n_3, n_4, n_6, n_7, n_8) = I_a(n_1, n_2, n_3, n_4, 0, n_6, n_7, n_8)$ (Fig. 12b). This integral is mirror-symmetric; it vanishes when the indices of the following groups of lines are non-positive: 12, 67, 68, 78, 37, 48, 13, 17, 24, 28, or 364. If any index is zero, the integral becomes trivial. If $n_1 < 0$, $n_2 < 0$, $n_3 < 0$, $n_4 < 0$, $n_6 < 0$, $n_7 < 0$, or $n_8 < 0$, we just consider $J_b$ as $I_a$ with $n_5 = 0$ and proceed as usual; the integral reduces to trivial ones not including $J_b$ (though, possibly, including $J_a$). When all the indices are positive, one of the lines 3, 4, 6 can be killed by (43).

## 3.2 Mercedez with three heavy-quark lines

Now let's consider the diagram of Fig. 11b. We define

$$\begin{aligned}
&\int \frac{N^{n_0} d^d k_1 d^d k_2 d^d k_3}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7} D_8^{n_8}} \\
&= -i\pi^{3d/2} (-2\omega)^{3d+2n_0-2\sum_{i=4}^{8} n_i} I_b(n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8; n_0)\,, \\
&D_1 = (k_1 + p) \cdot v/\omega\,, \quad D_2 = (k_2 + p) \cdot v/\omega\,, \quad D_3 = (k_3 + p) \cdot v/\omega\,, \\
&D_4 = -k_1^2\,, \quad D_5 = -k_2^2\,, \quad D_6 = -(k_1 - k_3)^2\,, \\
&D_7 = -(k_2 - k_3)^2\,, \quad D_8 = -(k_1 - k_2)^2\,, \quad N = -k_3^2\,.
\end{aligned} \tag{48}$$

This integral is mirror-symmetric with respect to $1 \leftrightarrow 2$, $4 \leftrightarrow 5$, $6 \leftrightarrow 7$. It vanishes when the indices of the following groups of lines are non-positive: 36, 37, 67, 148, 168, 416, 486, 258, 278, 527, or 587.

First we are going to get rid of the numerator. When $n_4 \neq 1$, we can lower $n_0$ by the $\partial_1 \cdot (k_1 - k_3)$ relation

$$\begin{aligned}
&\Big[d - n_1 - n_4 - n_8 - 2n_6 + n_1 \mathbf{1}^+ \mathbf{3}^- \\
&\quad + n_4 \mathbf{4}^+ (\mathbf{0}^+ - \mathbf{6}^-) + n_8 \mathbf{8}^+ (\mathbf{7}^- - \mathbf{6}^-)\Big] I_b = 0\,;
\end{aligned} \tag{49}$$

the case $n_5 \neq 0$ is symmetric. When $n_6 \neq 1$, we can lower $n_0$ by the $\partial_1 \cdot k_1$ relation

$$\Big[d - n_1 - n_6 - n_8 - 2n_4 + n_1 \mathbf{1}^+ + n_6 \mathbf{6}^+ (\mathbf{0}^+ - \mathbf{4}^-) + n_8 \mathbf{8}^+ (\mathbf{5}^- - \mathbf{4}^-)\Big] I_b = 0\,; \tag{50}$$

the case $n_7 \neq 1$ is symmetric. When $n_3 \neq 1$, we can lower $n_0$ or raise $n_6$ or $n_7$ by the $\partial_3 \cdot v$ relation

$$\Big[-2n_3 \mathbf{3}^+ + n_0 \mathbf{0}^- (1 - \mathbf{3}^-) + n_6 \mathbf{6}^+ (\mathbf{3}^- - \mathbf{1}^-) + n_7 \mathbf{7}^+ (\mathbf{3}^- - \mathbf{2}^-)\Big] I_b = 0\,. \tag{51}$$

Finally, we can lower $n_0$ or raise $n_3$ or $n_7$ by the $\partial_3 \cdot (k_3 - k_1)$ relation

$$\begin{aligned}
&\Big[d + n_0 - n_3 - n_7 - 2n_6 + n_3 \mathbf{3}^+ \mathbf{1}^- \\
&\quad + n_0 \mathbf{0}^- (\mathbf{6}^- - \mathbf{4}^-) + n_7 \mathbf{7}^+ (\mathbf{8}^- - \mathbf{6}^-)\Big] I_b = 0\,.
\end{aligned} \tag{52}$$

Now we shall discuss the integral $I_b(n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8)$ without numerator ($n_0 = 0$). Applying $\partial_1 \cdot v$, $\partial_1 \cdot (k_1 - k_2)$, $(\partial_1 + \partial_2 + \partial_3) \cdot k_1$ to the integrand of (48), we obtain the recurrence relations

$$\Big[-2n_1 \mathbf{1}^+ + n_4 \mathbf{4}^+ (\mathbf{1}^- - 1) + n_6 \mathbf{6}^+ (\mathbf{1}^- - \mathbf{3}^-) + n_8 \mathbf{8}^+ (\mathbf{1}^- - \mathbf{2}^-)\Big] I_b = 0\,, \tag{53}$$

$$\Big[d - n_1 - n_4 - n_6 - 2n_8 + n_1 \mathbf{1}^+ \mathbf{2}^-$$

$$+ n_4 \mathbf{4}^+(\mathbf{5}^- - \mathbf{8}^-) + n_6 \mathbf{6}^+(\mathbf{7}^- - \mathbf{8}^-)\Big] I_b = 0\,, \tag{54}$$

$$\Big[2(d - n_6 - n_7 - n_8) - n_2 - n_3 - n_5 + (n_2 \mathbf{2}^+ + n_3 \mathbf{3}^+)\mathbf{1}^-$$
$$+ n_5 \mathbf{5}^+(\mathbf{4}^- - \mathbf{8}^-)\Big] I_b = 0\,, \tag{55}$$

where the last relation was simplified using the homogeneity relation

$$\Big[3d - n_1 - n_2 - n_3 - 2(n_4 + n_5 + n_6 + n_7 + n_8)$$
$$+ n_1 \mathbf{1}^+ + n_2 \mathbf{2}^+ + n_3 \mathbf{3}^+\Big] I_b = 0\,. \tag{56}$$

Using it to simplify the sum of (53), its symmetric and (51), we get

$$\Big[2\left[3d - n_1 - n_2 - n_3 - 2(n_4 + n_5 + n_6 + n_7 + n_8)\right]$$
$$+ n_4 \mathbf{4}^+(\mathbf{1}^- - 1) + n_5 \mathbf{5}^+(\mathbf{2}^- - 1)\Big] I_b = 0\,. \tag{57}$$

The cases $n_3 = 0$, $n_4 = 0$, $n_5 = 0$, $n_6 = 0$, $n_7 = 0$ are trivial. When $n_3 < 0$ and $n_6 \neq 1$, we can raise $n_3$ by (53) (the case $n_7 \neq 1$ is symmetric); when $n_3 < 0$ and $n_6 = n_7 = 1$, we can raise $n_3$ or $n_7$ by (52). When $n_4 < 0$ and $n_5 \neq 1$, we can raise $n_4$ by the relation symmetric to (54); when $n_4 < 0$ and $n_5 = 1$, we can raise $n_4$ or $n_5$ by (57). The case $n_5 < 0$ is symmetric. When $n_6 < 0$ and $n_7 \neq 1$, we can raise $n_6$ by (52); when $n_6 < 0$ and $n_3 \neq 1$, $n_7 = 1$, we can raise $n_6$ or $n_7$ by (51); when $n_6 < 0$ and $n_3 = n_7 = 1$, we can raise $n_6$ or $n_3$ by the relation symmetric to (52). The case $n_7 < 0$ is symmetric. The case $n_1 = 0$ is $J_a(n_3, n_2, n_5, n_7, n_4, n_6, n_8)$ (Fig. 12a, Sect. 3.1), $n_2 = 0$ is symmetric. When $n_1 < 0$ and $n_8 \neq 1$, we can raise $n_1$ by the relation symmetric to (53); when $n_1 < 0$ and $n_6 \neq 1$, we can raise $n_1$ by (51); when $n_1 < 0$ and $n_4 \neq 1$, we can raise $n_1$ by (57); when $n_1 < 0$ and $n_2 \neq 1$, we can raise $n_1$ by the relation symmetric to (54); when $n_1 < 0$ and $n_3 \neq 1$, we can raise $n_1$ by (52); when $n_1 < 0$ and $n_2 = n_3 = n_4 = n_6 = n_8 = 1$, we can raise $n_1$, $n_2$ or $n_3$ by (56). The case $n_2 < 0$ is symmetric. The case $n_8 = 0$ (Fig. 12c, $J_c(n_1, n_2, n_3, n_4, n_5, n_6, n_7)$) will be considered later in this Section. When $n_8 < 0$ and $n_7 \neq 1$, we can raise $n_8$ by (52) (the case $n_6 \neq 1$ is symmetric); when $n_8 < 0$ and $n_5 \neq 1$, we can raise $n_8$ by (55) (the case $n_4 \neq 1$ is symmetric); when $n_8 < 0$ and $n_1 \neq 1$, we can raise $n_8$ by (53) (the case $n_2 \neq 1$ is symmetric); when $n_8 < 0$ and $n_3 \neq 1$, we can raise $n_8$ by (51); when $n_8 < 0$ and $n_1 = n_2 = n_3 = n_4 = n_5 = n_6 = n_7 = 1$, we can raise $n_1$, $n_2$ or $n_3$ by (56). When all the indices are positive, we can kill one of the lines 1, 6, 8 by (52), or one of the lines 2, 7, 8 by its mirror-symmetric relation.

Now we consider $J_c(n_1, n_2, n_3, n_4, n_5, n_6, n_7) = I_b(n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8)$ (Fig. 12c). It is mirror-symmetric; it vanishes if any two indices of $n_4$,

$n_5$, $n_6$, $n_7$ are non-positive, or $n_1$, $n_2$, $n_3$ are all non-positive. If any index is zero, the integral becomes trivial. Using the $\partial_1 \cdot v$ and $\partial_3 \cdot v$ relations

$$\left[-2n_1\mathbf{1}^+ + n_4\mathbf{4}^+(\mathbf{1}^- - 1) + n_6\mathbf{6}^+(\mathbf{1}^- - \mathbf{3}^-)\right] J_c = 0\,, \tag{58}$$

$$\left[-2n_3\mathbf{3}^+ + n_6\mathbf{6}^+(\mathbf{3}^- - \mathbf{1}^-) + n_7\mathbf{7}^+(\mathbf{3}^- - \mathbf{2}^-)\right] J_c = 0\,, \tag{59}$$

we can lower $n_1$, $n_2$ (symmetric case) and $n_3$ down to 1.

We are left with $J_c(1,1,1,n_4,n_5,n_6,n_7)$. It is rather difficult to apply the standard techniques to this integral, because each operator $\partial_i \cdot k_j$ produces a scalar product which cannot be expressed via the denominators. For example, $\partial_1 \cdot k_1$ gives the term $-\frac{n_6}{D_6} 2k_1 \cdot k_3$, and $\partial_1 \cdot (k_1 - k_3)$ gives the term $-\frac{n_4}{D_4} 2k_1 \cdot k_3$. We can cancel $2k_1 \cdot k_3$ by forming the difference

$$(n_4 - 1)\mathbf{6}^- \partial_1 \cdot k_1 - (n_6 - 1)\mathbf{4}^- \partial_1 \cdot (k_1 - k_3)\,,$$

which results in

$$\begin{aligned}&\Big[(n_4 - 1)\left(d - n_1 - 2n_4 + n_1\mathbf{1}^+\right)\mathbf{6}^- \\ &\quad - (n_6 - 1)\left(d - n_1 - 2n_6 + n_1\mathbf{1}^+\mathbf{3}^-\right)\mathbf{4}^-\Big] J_c = 0\,.\end{aligned}$$

Some other useful combinations are

$$\begin{aligned}&(n_6 - 1)\mathbf{7}^- \partial_3 \cdot (k_3 - k_1) - (n_7 - 1)\mathbf{6}^- \partial_3 \cdot (k_3 - k_2)\,,\\ &(n_4 - 1)\mathbf{5}^- (\partial_1 + \partial_2 + \partial_3) \cdot k_1 - (n_5 - 1)\mathbf{4}^- (\partial_1 + \partial_2 + \partial_3) \cdot k_2\,,\\ &(n_5 - 1)\mathbf{6}^- \partial_3 \cdot (k_3 - k_2) - (n_6 - 1)\mathbf{5}^- (\partial_1 + \partial_2 + \partial_3) \cdot k_1\,,\\ &(n_4 - 1)\mathbf{6}^- \partial_3 \cdot (k_3 - k_2) - (n_6 - 1)\mathbf{4}^- (\partial_1 + \partial_2 + \partial_3) \cdot k_2\,,\\ &(n_4 - 1)\mathbf{7}^- (\partial_1 + \partial_3) \cdot k_1 - (n_7 - 1)\mathbf{4}^- (\partial_1 + \partial_3) \cdot (k_3 - k_2)\,,\end{aligned}$$

where all $\partial_1 + \partial_2 + \partial_3$ pieces can be simplified by the homogeneity relation. Using (58), (59), we obtain at $n_1 = n_2 = n_3 = 1$

$$\begin{aligned}&\Big[(n_4 - 1)\left[2(d - 2n_4 - 1) - n_4\mathbf{4}^+(1 - \mathbf{1}^-)\right]\mathbf{6}^- \\ &\quad - 2(n_6 - 1)(d - 2n_6 - 1 - \mathbf{1}^+\mathbf{3}^-)\mathbf{4}^- \\ &\quad + (n_4 - 1)(n_6 - 1)(\mathbf{1}^- - \mathbf{3}^-)\Big] J_c = 0\,,\end{aligned} \tag{60}$$

$$\begin{aligned}&[(n_6 - 1)(d - 2n_6 - 1 + \mathbf{3}^+\mathbf{1}^-)\mathbf{7}^- \\ &\quad - (n_7 - 1)(d - 2n_7 - 1 + \mathbf{3}^+\mathbf{2}^-)\mathbf{6}^-] J_c = 0\,,\end{aligned} \tag{61}$$

$$\begin{aligned}&\Big[(n_4 - 1)\left[2(d - n_5 - n_6 - n_7) + (\mathbf{2}^+ + \mathbf{3}^+)\mathbf{1}^-\right]\mathbf{5}^- \\ &\quad - (n_5 - 1)\left[2(d - n_4 - n_6 - n_7) + (\mathbf{1}^+ + \mathbf{3}^+)\mathbf{2}^-\right]\mathbf{4}^-\Big] J_c = 0\,,\end{aligned} \tag{62}$$

$$
\begin{aligned}
&\Big[(n_5-1)(d-2n_7-n_6+\mathbf{3}^+\mathbf{2}^-)\mathbf{6}^- \\
&\quad +(n_6-1)\left[2(d-n_6-n_7)-n_5-1+(\mathbf{2}^++\mathbf{3}^+)\mathbf{1}^-\right]\mathbf{5}^- \\
&\quad +(n_5-1)(n_6-1)(\mathbf{4}^--\mathbf{7}^-)\Big]J_c=0\,, \qquad (63)\\
&\Big[(n_4-1)(d-2n_7-n_6+\mathbf{3}^+\mathbf{2}^-)\mathbf{6}^- \\
&\quad +(n_6-1)\left[2(d-n_6-n_7)-n_4-1+(\mathbf{4}^++\mathbf{3}^+)\mathbf{2}^-\right]\mathbf{4}^- \\
&\quad +(n_4-1)(n_6-1)(\mathbf{5}^--\mathbf{7}^-)\Big]J_c=0\,, \qquad (64)\\
&\Big[(n_4-1)[2(d-2n_4-1-\mathbf{3}^+\mathbf{1}^-)-n_4\mathbf{4}^+(1-\mathbf{1}^-) \\
&\quad +(n_7-1)\mathbf{6}^+(\mathbf{3}^--\mathbf{2}^-)]\mathbf{7}^- \\
&\quad -2(n_7-1)\left[d-2n_7-1+\mathbf{1}^+\mathbf{3}^--(\mathbf{1}^++\mathbf{3}^+)\mathbf{2}^-\right]\mathbf{4}^-\Big]J_c=0\,. \qquad (65)
\end{aligned}
$$

One more useful relation is obtained by adding (58), its mirror-symmetric, formula (59) and using the homogeneity relation:

$$
\begin{aligned}
&\Big[2\left[3d-n_1-n_2-n_3-2(n_4+n_5+n_6+n_7)\right] \\
&\quad -n_4\mathbf{4}^+(1-\mathbf{1}^-)-n_5\mathbf{5}^+(1-\mathbf{2}^-)\Big]J_c=0\,. \qquad (66)
\end{aligned}
$$

When $n_4<0$ and $n_6\neq 1$, we can raise $n_4$ by (60); when $n_4<0$ and $n_5\neq 1$, we can raise $n_4$ by (62); when $n_4<0$ and $n_7\neq 1$, we can raise $n_4$ by (65); when $n_4<0$ and $n_5=n_6=n_7=1$, we can raise $n_4$ or $n_5$ by (66). The case $n_5<0$ is symmetric. When $n_6<0$ and $n_7\neq 1$, we can raise $n_6$ by (61); when $n_6<0$ and $n_4\neq 1$, we can raise $n_6$ by (64); when $n_6<0$ and $n_5\neq 1$, we can raise $n_4$ by (63); when $n_6<0$ and $n_4=n_5=n_7=1$, we can raise $n_4$ or $n_5$ by (66). The case $n_7<0$ is symmetric. When all the indices are positive, we can lower $n_6$ down to 1 (raising $n_4$) by (60) and lower $n_7$ down to 1 (raising $n_5$) by its mirror-symmetric. Finally, we can lower $n_4$ and $n_5$ down to 1 by (62) together with (66).

The integral $J_c(1,1,1,1,1,1,1)$ cannot be reduced to simpler ones (in contrast to the massless case [3]), and should be considered as a new basis integral. Its value is currently unknown, and its calculation is highly non-trivial.

### 3.3 Ladder with three heavy-quark lines

Now let's consider the diagram of Fig. 11c. We define

$$
\int \frac{N^{n_0} d^dk_1 d^dk_2 d^dk_3}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7} D_8^{n_8}}
$$

$$
\begin{aligned}
&= -i\pi^{3d/2}(-2\omega)^{3d+2n_0-2\sum_{i=4}^{8} n_i} I_c(n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8;n_0)\,,\\
&D_1 = (k_1+p)\cdot v/\omega\,,\quad D_2 = (k_2+p)\cdot v/\omega\,,\quad D_3 = (k_3+p)\cdot v/\omega\,,\\
&D_4 = -k_1^2\,,\quad D_5 = -k_2^2\,,\quad D_6 = -(k_1-k_3)^2\,,\\
&D_7 = -(k_2-k_3)^2\,,\quad D_8 = -k_3^2\,,\quad N = 2k_1\cdot k_2\,.
\end{aligned}
\tag{67}
$$

This integral is mirror-symmetric with respect to $1 \leftrightarrow 2$, $4 \leftrightarrow 5$, $6 \leftrightarrow 7$. It vanishes when the indices of the following groups of lines are non-positive: 14, 16, 46, 25, 27, 57, 132, 637, 368, 378, 687, 485, 487, 586.

First we are going to get rid of the numerator. When $n_7 \neq 1$, we can lower $n_0$ by the $\partial_3 \cdot (k_3 - k_1)$ relation

$$
\begin{aligned}
&\Big[d - n_3 - n_7 - n_8 - 2n_6 + n_3\mathbf{3}^+\mathbf{1}^-\\
&\quad + n_7\mathbf{7}^+(\mathbf{0}^+ + \mathbf{4}^- + \mathbf{5}^- - \mathbf{7}^-) + n_8\mathbf{8}^+(\mathbf{5}^- - \mathbf{7}^-)\Big] I_c = 0\,;
\end{aligned}
\tag{68}
$$

the case $n_6 \neq 0$ is symmetric. When $n_5 \neq 1$, we can lower $n_0$ by

$$
\begin{aligned}
&\Big[2(d - n_5 - n_6 - n_7) + n_0 - n_2 - n_3 - n_8 + 2n_0\mathbf{0}^-\mathbf{4}^- + (n_2\mathbf{2}^+ + n_3\mathbf{3}^+)\mathbf{1}^-\\
&\quad - n_5\mathbf{5}^+\mathbf{0}^+ + n_8\mathbf{8}^+(\mathbf{4}^- - \mathbf{6}^-)\Big] I_c = 0\,,
\end{aligned}
\tag{69}
$$

which is the $(\partial_1+\partial_2+\partial_3)\cdot k_1$ relation simplified by the homogeneity relation. the case $n_4 \neq 1$ is symmetric. When $n_1 \neq 1$, we can lower $n_0$ or raise $n_4$ or $n_6$ by the $\partial_1 \cdot v$ relation

$$
\Big[-2n_1\mathbf{1}^+ + n_0\mathbf{0}^-(\mathbf{2}^- - 1) + n_4\mathbf{4}^+(\mathbf{1}^- - 1) + n_6\mathbf{6}^+(\mathbf{1}^- - \mathbf{3}^-)\Big] I_c = 0\,; \tag{70}
$$

the case $n_2 \neq 1$ is symmetric. Finally, we can raise $n_1$ or $n_6$ by the $\partial_1 \cdot k_1$ relation

$$
\Big[d + n_0 - n_1 - n_6 - 2n_4 + n_1\mathbf{1}^+ + n_6\mathbf{6}^+(\mathbf{8}^- - \mathbf{4}^-)\Big] I_c = 0\,. \tag{71}
$$

Now we shall discuss the integral $I_c(n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8)$ without numerator ($n_0 = 0$). Applying $\partial_3 \cdot v$, $\partial_1 \cdot (k_1 - k_3)$ to the integrand of (67), we obtain the recurrence relations

$$
\Big[-2n_3\mathbf{3}^+ + n_6\mathbf{6}^+(\mathbf{1}^- - \mathbf{3}^-) + n_7\mathbf{7}^+(\mathbf{2}^- - \mathbf{3}^-) + n_8\mathbf{8}^+(1 - \mathbf{3}^-)\Big] I_c = 0\,, \tag{72}
$$

$$
\Big[d - n_1 - n_4 - 2n_6 + n_1\mathbf{1}^+\mathbf{3}^- + n_4\mathbf{4}^+(\mathbf{8}^- - \mathbf{6}^-)\Big] I_c = 0\,. \tag{73}
$$

Homogeneity in $\omega$ gives the relation identical with (56).

The cases $n_1 = 0$, $n_2 = 0$, $n_4 = 0$, $n_5 = 0$, $n_6 = 0$, $n_7 = 0$ are trivial. When $n_1 < 0$ and $n_6 \neq 1$, we can raise $n_1$ by (72); when $n_1 < 0$ and $n_4 \neq 1$, we can raise $n_1$ using the sum of (70) and (72); when $n_1 < 0$ and $n_4 = n_6 = 1$, we can raise $n_1$ or $n_6$ by (71). The case $n_2 < 0$ is symmetric. When $n_4 < 0$ and $n_6 \neq 1$, we can raise $n_4$ by (71); when $n_4 < 0$ and $n_1 \neq 1$, $n_6 = 1$, we can raise $n_4$ or $n_6$ by (70); when $n_4 < 0$ and $n_1 = n_6 = 1$, we can raise $n_4$ or $n_1$ by (73). The case $n_5 < 0$ is symmetric. When $n_6 < 0$ and $n_4 \neq 1$, we can raise $n_6$ by (73); when $n_6 < 0$ and $n_1 \neq 1$, $n_4 = 1$, we can raise $n_6$ or $n_4$ by (70); when $n_6 < 0$ and $n_1 = n_4 = 1$, we can raise $n_6$ or $n_1$ by (71). The case $n_7 < 0$ is symmetric. The case $n_3 = 0$ is $J_b(n_1, n_2, n_4, n_5, n_8, n_6, n_7)$ (Fig. 12a, Sect. 3.1). When $n_3 < 0$ and $n_6 \neq 1$, we can raise $n_3$ by (70) (the case $n_7 \neq 1$ is symmetric); when $n_3 < 0$ and $n_8 \neq 1$, we can raise $n_3$ by the relation

$$\begin{aligned}&\Big[2\left[3d - n_1 - n_2 - n_3 - 2(n_4 + n_5 + n_6 + n_7 + n_8)\right]\\&\quad + n_4\mathbf{4}^+(\mathbf{1}^- - 1) + n_5\mathbf{5}^+(\mathbf{2}^- - 1) + n_8\mathbf{8}^+(\mathbf{3}^- - 1)\Big] I_c = 0\,,\end{aligned} \tag{74}$$

which is (70) plus its mirror-symmetric plus (72) simplified by the homogeneity relation; when $n_3 < 0$ and $n_1 \neq 1$, we can raise $n_3$ by (73) (the case $n_2 \neq 1$ is symmetric); when $n_3 < 0$ and $n_1 = n_2 = n_6 = n_8 = 1$, we can raise $n_1$, $n_2$ or $n_3$ by the homogeneity relation. The case $n_8 = 0$ is $J_c(n_1, n_2, n_3, n_4, n_5, n_6, n_7)$ (Fig. 12c, Sect. 3.2). When $n_8 < 0$ and $n_6 \neq 1$, we can raise $n_8$ by (71) (the case $n_7 \neq 1$ is symmetric); when $n_8 < 0$ and $n_4 \neq 1$, we can raise $n_8$ by (73) (the case $n_5 \neq 1$ is symmetric); when $n_8 < 0$ and $n_3 \neq 1$, $n_4 = n_6 = n_7 = 1$, we can raise $n_8$, $n_6$ or $n_7$ by (72); when $n_8 < 0$ and $n_1 \neq 1$, $n_3 = n_4 = n_6 = n_7 = 1$, we can raise $n_4$ or $n_6$ by (70) (the case $n_2 \neq 1$ is symmetric); when $n_8 < 0$ and $n_1 = n_2 = n_3 = n_4 = n_5 = n_6 = n_7 = 1$, we can raise $n_1$, $n_2$ or $n_3$ the homogeneity relation. When all the indices are positive, we can kill one of the lines 3, 6, 8 by (73), or one of the lines 3, 7, 8 by its mirror-symmetric relation.

### 3.4 Non-planar diagram with three heavy-quark lines

Now let's consider the diagram of Fig. 11d. We define

$$\begin{aligned}&\int \frac{N^{n_0} d^d k_1 d^d k_2 d^d k_3}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7} D_8^{n_8}}\\&= -i\pi^{3d/2}(-2\omega)^{3d+2n_0-2\sum_{i=4}^{8} n_i} I_d(n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8; n_0)\,,\\&D_1 = (k_1 + p)\cdot v/\omega\,,\quad D_2 = (k_2 + p)\cdot v/\omega\,,\\&D_3 = (k_1 + k_2 - k_3 + p)\cdot v/\omega\,,\quad D_4 = -k_1^2\,,\quad D_5 = -k_2^2\,,\\&D_6 = -(k_1 - k_3)^2\,,\quad D_7 = -(k_2 - k_3)^2\,,\quad D_8 = -k_3^2\,,\quad N = 2k_1\cdot k_2\,.\end{aligned} \tag{75}$$

This integral is mirror-symmetric with respect to $1 \leftrightarrow 2$, $4 \leftrightarrow 5$, $6 \leftrightarrow 7$. It vanishes when the indices of the following groups of lines are non-positive:

46, 57, 132, 485, 487, 586, 687, 314, 136, 325, 237, 148, 178, 258, 268, 417, 526, 637, 368, 378.

Applying $\partial_1 \cdot v$, $\partial_3 \cdot v$, $\partial_1 \cdot k_1$, $\partial_1 \cdot (k_1 - k_3)$, $(\partial_1 + \partial_3) \cdot k_3$, $\partial_3 \cdot (k_3 - k_1)$, $(\partial_1 + \partial_2 + \partial_3) \cdot k_1$ to the integrand of (75), we obtain the recurrence relations

$$\begin{aligned}&\Big[n_0 \mathbf{0}^-(\mathbf{2}^- - 1) - 2n_1 \mathbf{1}^+ - 2n_3 \mathbf{3}^+\\&\quad + n_4 \mathbf{4}^+(\mathbf{1}^- - 1) + n_6 \mathbf{6}^+(\mathbf{3}^- - \mathbf{2}^-)\Big] I_d = 0\,,\end{aligned} \tag{76}$$

$$\begin{aligned}&\Big[2n_3 \mathbf{3}^+ + n_6 \mathbf{6}^+(\mathbf{2}^- - \mathbf{3}^-) + n_7 \mathbf{7}^+(\mathbf{1}^- - \mathbf{3}^-)\\&\quad + n_8 \mathbf{8}^+(\mathbf{1}^- + \mathbf{2}^- - \mathbf{3}^- - 1)\Big] I_d = 0\,,\end{aligned} \tag{77}$$

$$\begin{aligned}&\Big[d + n_0 - n_1 - n_6 - 2n_4 + n_1 \mathbf{1}^+ + n_3 \mathbf{3}^+(1 - \mathbf{1}^-)\\&\quad + n_6 \mathbf{6}^+(\mathbf{8}^- - \mathbf{4}^-)\Big] I_d = 0\,,\end{aligned} \tag{78}$$

$$\begin{aligned}&\Big[d + n_0 - n_3 - n_4 - 2n_6 + n_0 \mathbf{0}^-(\mathbf{5}^- - \mathbf{7}^- + \mathbf{8}^-)\\&\quad + n_1 \mathbf{1}^+(\mathbf{2}^- - \mathbf{3}^-) + n_3 \mathbf{3}^+ \mathbf{2}^- + n_4 \mathbf{4}^+(\mathbf{8}^- - \mathbf{6}^-)\Big] I_d = 0\,,\end{aligned} \tag{79}$$

$$\begin{aligned}&\Big[d - n_1 - n_4 - n_7 - 2n_8 + n_0 \mathbf{0}^-(\mathbf{7}^- - \mathbf{5}^- - \mathbf{8}^-)\\&\quad + n_1 \mathbf{1}^+(1 - \mathbf{2}^- + \mathbf{3}^-) + n_4 \mathbf{4}^+(\mathbf{6}^- - \mathbf{8}^-) + n_7 \mathbf{7}^+(\mathbf{5}^- - \mathbf{8}^-)\Big] I_d = 0\,,\end{aligned} \tag{80}$$

$$\begin{aligned}&\Big[d - n_3 - n_7 - n_8 - 2n_6 + n_3 \mathbf{3}^+ \mathbf{2}^-\\&\quad + n_7 \mathbf{7}^+(\mathbf{0}^+ + \mathbf{4}^- + \mathbf{5}^- - \mathbf{6}^-) + n_8 \mathbf{8}^+(\mathbf{4}^- - \mathbf{6}^-)\Big] I_d = 0\,,\end{aligned} \tag{81}$$

$$\begin{aligned}&\Big[2(d - n_5 - n_6 - n_7) + n_0 - n_2 - n_3 - n_8 + 2n_0 \mathbf{0}^- \mathbf{4}^- + (n_2 \mathbf{2}^+ + n_3 \mathbf{3}^+)\mathbf{1}^-\\&\quad - n_5 \mathbf{5}^+ \mathbf{0}^+ + n_8 \mathbf{8}^+(\mathbf{4}^- - \mathbf{6}^-)\Big] I_d = 0\,,\end{aligned} \tag{82}$$

Homogeneity in $\omega$ gives

$$\begin{aligned}&\Big[3d - n_1 - n_2 - n_3 - 2(n_4 + n_5 + n_6 + n_7 + n_8 - n_0)\\&\quad + n_1 \mathbf{1}^+ + n_2 \mathbf{2}^+ + n_3 \mathbf{3}^+\Big] I_d = 0\end{aligned} \tag{83}$$

(this was used to simplify (82)).

The cases

$$\begin{aligned}&I_d(0, n_2, n_3, n_4, n_5, n_6, n_7, n_8; n_0) = (\mathbf{5}^- + \mathbf{6}^- - \mathbf{3}^- - \mathbf{8}^-)^{n_0}\\&\quad \times I_a(n_3, n_2, 0, n_5, n_6, n_7, n_4, n_8)\,,\end{aligned} \tag{84}$$

$$\begin{aligned}&I_d(n_1, n_2, 0, n_4, n_5, n_6, n_7, n_8; n_0) = (\mathbf{5}^- - \mathbf{3}^- - \mathbf{4}^-)^{n_0}\\&\quad \times I_a(n_1, n_2, n_4, n_5, 0, n_8, n_6, n_7)\end{aligned} \tag{85}$$

($n_2 = 0$ is symmetric to $n_1 = 0$) reduce to $I_a$ (Fig. 11a, Sect. 3.1). When $n_1 < 0$ and $n_4 \neq 1$, we can raise $n_1$ by (76); when $n_1 < 0$ and $n_7 \neq 1$, we can raise $n_1$ by the relation symmetric to (76); when $n_1 < 0$ and $n_8 \neq 1$, we can raise $n_1$ using the sum of (77) and the relation symmetric to (76); when $n_1 < 0$ and $n_3 \neq 1$, we can raise $n_1$ by (78); when $n_1 < 0$ and $n_2 \neq 1$, we can raise $n_1$ by the relation symmetric to (80); when $n_1 < 0$ and $n_2 = n_3 = n_4 = n_7 = n_8 = 1$, we can raise $n_1$, $n_2$ or $n_3$ by (83). The case $n_2 < 0$ is symmetric. When $n_3 < 0$ and $n_6 \neq 1$, we can raise $n_3$ by (76) (the case $n_7 \neq 1$ is symmetric); when $n_3 < 0$ and $n_8 \neq 1$, we can raise $n_3$ using the sum of (76), its symmetric and (77); when $n_3 < 0$ and $n_1 \neq 1$, we can raise $n_3$ by (79) (the case $n_2 \neq 1$ is symmetric); when $n_3 < 0$ and $n_1 = n_2 = n_6 = n_7 = n_8 = 1$, we can raise $n_1$, $n_2$ or $n_3$ by (83). When $n_1 > 1$, we can lower it using the sum of (76) and (77); the case $n_2 > 1$ is symmetric. When $n_3 > 1$, we can lower it by (77).

We are left with $I_d(1, 1, 1, n_4, n_5, n_6, n_7, n_8; n_0)$. Now we are going to get rid of the numerator. When $n_7 \neq 1$, we can lower $n_0$ by (81); the case $n_6 \neq 0$ is symmetric. When $n_5 \neq 1$, we can lower $n_0$ by (82); the case $n_4 \neq 1$ is symmetric. Otherwise, we use (79) to raise $n_4$, plus trivial terms.

Now we shall discuss the integral $I_d(1, 1, 1, n_4, n_5, n_6, n_7, n_8)$ without numerator ($n_0 = 0$). Using (76), (77), we obtain from (83), (78), (79) at $n_1 = n_2 = n_3 = 1$

$$\Big[2[3(d-1) - 2(n_4 + n_5 + n_6 + n_7 + n_8)] - n_4\mathbf{4}^+(1 - \mathbf{1}^-) - n_5\mathbf{5}^+(1 - \mathbf{2}^-) - n_8\mathbf{8}^+(1 - \mathbf{1}^- - \mathbf{2}^- + \mathbf{3}^-)\Big] I_d = 0\,, \tag{86}$$

$$\Big[2(d - n_6 - 2n_4 - 1) - n_4\mathbf{4}^+(1 - \mathbf{1}^-) + n_6\mathbf{6}^+(2(\mathbf{8}^- - \mathbf{4}^-) + \mathbf{3}^- - \mathbf{2}^-) - 2\mathbf{3}^+\mathbf{1}^-\Big] I_d = 0\,, \tag{87}$$

$$\Big[d - n_4 - 2n_6 - 1 + n_4\mathbf{4}^+(\mathbf{8}^- - \mathbf{6}^-) + \mathbf{1}^+(\mathbf{2}^- - \mathbf{3}^-)\Big] I_d = 0\,. \tag{88}$$

The case $n_8 = 0$ ($J_c(n_1, n_2, n_3, n_4, n_5, n_7, n_6)$, Fig. 12c, Sect. 3.2) was discussed in Sect. 3.2; the cases $n_6 = 0$ ($J_d(n_1, n_3, n_2, n_4, n_7, n_5, n_8)$, Fig. 12d; $n_7 = 0$ is symmetric) and $n_4 = 0$ ($J_e(n_1, n_3, n_2, n_6, n_8, n_5, n_7)$, Fig. 12e; $n_5 = 0$ is symmetric) will be discussed in Sect. 3.5. If $n_4 < 0$, it can be raised by (88); the case $n_5 < 0$ is symmetric. When $n_6 < 0$ and $n_4 \neq 1$, we can raise $n_6$ by (88); when $n_6 < 0$ and $n_4 = 1$, we can raise $n_6$ or $n_4$ by (87). The case $n_7 < 0$ is symmetric. When $n_8 < 0$ and $n_4 \neq 1$, we can raise $n_8$ by (88) (the case $n_5 \neq 1$ is symmetric); when $n_8 < 0$ and $n_6 \neq 1$, we can raise $n_8$ by (87) (the case $n_7 \neq 1$ is symmetric); when $n_8 < 0$ and $n_4 = n_5 = n_6 = n_7 = 1$, we can raise $n_8$, $n_4$ or $n_5$ by (86). When the indices are positive, we can kill the line 6 or 8 using (88).

Thus, the non-planar diagram reduces to planar ones, in contrast to the massless case where such a reduction is impossible [3, 21].

### 3.5 Diagrams with four heavy-quark lines

Let's define (Fig. 11e)

$$
\begin{aligned}
&\int \frac{N_{13}^{n_{13}} N_{23}^{n_{23}} d^d k_1 d^d k_2 d^d k_3}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7} D_8^{n_8}}\\
&= -i\pi^{3d/2} (-2\omega)^{3d+2(n_{13}+n_{23})-2\sum_{i=5}^{8} n_i} I_e(n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8;n_{13},n_{23})\,,\\
&D_1 = (k_3+p)\cdot v/\omega\,,\quad D_2 = (k_1+k_3+p)\cdot v/\omega\,,\\
&D_3 = (k_1+p)\cdot v/\omega\,,\quad D_4 = (k_2+p)\cdot v/\omega\,,\\
&D_5 = -k_3^2\,,\quad D_6 = -k_1^2\,,\quad D_7 = -k_2^2\,,\quad D_8 = -(k_1-k_2)^2\,,\\
&N_{13} = 2k_1\cdot k_3\,,\quad N_{23} = 2k_2\cdot k_3\,.
\end{aligned}
\tag{89}
$$

This integral vanishes when the indices of the following groups of lines are non-positive: 5, 12, 67, 68, 78, 47, 48, 234, 326, 238. The heavy-quark denominators are linearly dependent: $D_1 - D_2 + D_3 = 1$, and therefore

$$
\left[1 - \mathbf{1}^- + \mathbf{2}^- - \mathbf{3}^-\right] I_e = 0\,.
\tag{90}
$$

If $n_1 < 0$, it can be raised by (90); if $n_2 < 0$, it can be raised by (90); if $n_3 < 0$, it can be raised by (90). If $n_{1,2,3}$ are all positive, we can lower them by (90) until one of them vanish. The cases

$$
\begin{aligned}
&I_e(0,n_2,n_3,n_4,n_5,n_6,n_7,n_8;n_{13},n_{23}) = (\mathbf{6}^- - \mathbf{4}^- + \mathbf{8}^-)^{n_{13}}\\
&\quad\times(\mathbf{5}^- - \mathbf{7}^- + \mathbf{8}^- + \mathbf{0}^+)^{n_{23}} I_c(n_2,n_4,n_3,0,n_7,n_5,n_8,n_6)\,,\\
&I_e(n_1,0,n_3,n_4,n_5,n_6,n_7,n_8;n_{13},n_{23}) = (\mathbf{6}^- - \mathbf{4}^- - \mathbf{8}^-)^{n_{13}}\\
&\quad\times(\mathbf{0}^+)^{n_{23}} I_c(n_1,n_4,n_3,n_5,n_7,0,n_8,n_6)\,,\\
&I_e(n_1,n_2,0,n_4,n_5,n_6,n_7,n_8;n_{13},n_{23}) = (\mathbf{4}^- - \mathbf{6}^- + \mathbf{8}^- + \mathbf{0}^+)^{n_{13}}\\
&\quad\times(\mathbf{0}^+)^{n_{23}} I_d(n_1,n_4,n_2,n_5,n_7,0,n_6,n_8)
\end{aligned}
$$

reduce to $I_c$ (Fig. 11c, Sect. 3.3) and $I_d$ (Fig. 11d, Sect. 3.4).

Let's define (Fig. 11f)

$$
\begin{aligned}
&\int \frac{N_{13}^{n_{13}} N_{23}^{n_{23}} d^d k_1 d^d k_2 d^d k_3}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7} D_8^{n_8}}\\
&= -i\pi^{3d/2} (-2\omega)^{3d+2(n_{13}+n_{23})-2\sum_{i=5}^{8} n_i} I_f(n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8;n_{13},n_{23})\,,\\
&D_1 = (k_1+p)\cdot v/\omega\,,\quad D_2 = (k_1+k_3+p)\cdot v/\omega\,,\\
&D_3 = (k_2+k_3+p)\cdot v/\omega\,,\quad D_4 = (k_2+p)\cdot v/\omega\,,\\
&D_5 = -k_3^2\,,\quad D_6 = -k_1^2\,,\quad D_7 = -k_2^2\,,\quad D_8 = -(k_1-k_2)^2\,.
\end{aligned}
\tag{91}
$$

This integral is mirror-symmetric with respect to $1 \leftrightarrow 4$, $2 \leftrightarrow 3$, $6 \leftrightarrow 7$. It vanishes when indices of the following groups of lines are non-positive: 5, 23, 67, 68, 78, 216, 128, 347, 438. The heavy-quark denominators are linearly dependent: $D_1 - D_2 + D_3 - D_4 = 0$, and therefore

$$\left[\mathbf{1}^- - \mathbf{2}^- + \mathbf{3}^- - \mathbf{4}^-\right] I_f = 0\,. \tag{92}$$

If $n_1 < 0$, it can be raised by (92) (the case $n_4 = 0$ is symmetric); if $n_2 < 0$, it can be raised by (92) (the case $n_3 = 0$ is symmetric). Otherwise, we can use (92) to raise, say, $n_1$ and kill one of the lines 2, 3, 4. The cases

$$\begin{aligned}
&I_f(0, n_2, n_3, n_4, n_5, n_6, n_7, n_8; n_{13}, n_{23}) = (\mathbf{7}^- - \mathbf{5}^- + \mathbf{8}^-)^{n_{13}}\\
&\quad \times (\mathbf{8}^- - \mathbf{5}^- - \mathbf{7}^-)^{n_{23}} I_d(n_2, n_4, n_3, 0, n_7, n_5, n_8, n_6)\,,\\
&I_f(n_1, 0, n_3, n_4, n_5, n_6, n_7, n_8; n_{13}, n_{23}) = (\mathbf{8}^- - \mathbf{4}^- - \mathbf{6}^-)^{n_{13}}\\
&\quad \times (\mathbf{6}^- - \mathbf{4}^- + \mathbf{8}^-)^{n_{23}} I_c(n_1, n_3, n_4, n_6, 0, n_8, n_5, n_7)
\end{aligned}$$

($n_3 = 0$ is symmetric to $n_2 = 0$, $n_4 = 0$ is symmetric to $n_1 = 0$) reduce to $I_c$ (Fig. 11c, Sect. 3.3) and $I_d$ (Fig. 11d, Sect. 3.4).

Let's define (Fig. 11g)

$$\begin{aligned}
&\int \frac{N_{13}^{n_{13}} N_{23}^{n_{23}} d^d k_1 d^d k_2 d^d k_3}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7} D_8^{n_8}}\\
&= -i\pi^{3d/2} (-2\omega)^{3d+2(n_{13}+n_{23})-2\sum_{i=4}^{8} n_i} I_g(n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8; n_{13}, n_{23})\,,\\
&D_1 = (k_3 + p)\cdot v/\omega\,, \quad D_2 = (k_1 + k_3 + p)\cdot v/\omega\,,\\
&D_3 = (k_2 + k_3 + p)\cdot v/\omega\,, \quad D_4 = (k_2 + p)\cdot v/\omega\,,\\
&D_5 = -k_3^2\,, \quad D_6 = -k_1^2\,, \quad D_7 = -k_2^2\,, \quad D_8 = -(k_1 - k_2)^2\,.
\end{aligned} \tag{93}$$

This integral vanishes when the indices of the following groups of lines are non-positive: 5, 67, 68, 78, 26, 28, 123, 234, 347, 438. The heavy-quark denominators are linearly dependent: $D_1 - D_3 + D_4 = 1$, and therefore

$$\left[1 - \mathbf{1}^- + \mathbf{3}^- - \mathbf{4}^-\right] I_g = 0\,. \tag{94}$$

If $n_1 < 0$, it can be raised by (94); if $n_3 < 0$, it can be raised by (94); if $n_4 < 0$, it can be raised by (94). If $n_{1,3,4}$ are all positive, we can lower them by (94) until one of them vanish. The cases

$$\begin{aligned}
&I_g(0, n_2, n_3, n_4, n_5, n_6, n_7, n_8; n_{13}, n_{23}) = (\mathbf{6}^- - \mathbf{4}^- + \mathbf{8}^-)^{n_{13}}\\
&\quad \times (\mathbf{5}^- - \mathbf{7}^- + \mathbf{8}^- + \mathbf{0}^+)^{n_{23}} I_d(n_2, n_4, n_3, 0, n_7, n_5, n_8, n_6)\,,\\
&I_g(n_1, n_2, 0, n_4, n_5, n_6, n_7, n_8; n_{13}, n_{23}) = (\mathbf{4}^- - \mathbf{6}^- + \mathbf{8}^- + \mathbf{0}^+)^{n_{13}}\\
&\quad \times (\mathbf{0}^+)^{n_{23}} I_d(n_1, n_4, n_2, n_5, n_7, 0, n_6, n_8)\,,\\
&I_g(n_1, n_2, n_3, 0, n_5, n_6, n_7, n_8; n_{13}, n_{23}) = (\mathbf{4}^- + \mathbf{6}^- - \mathbf{0}^+)^{n_{13}}\\
&\quad \times (\mathbf{4}^- - \mathbf{5}^- + \mathbf{8}^-)^{n_{23}} I_b(n_1, n_3, n_2, n_5, 0, n_6, n_8, n_7)
\end{aligned}$$

reduce to $I_b$ (Fig. 11b, Sect. 3.2) and $I_d$ (Fig. 11d, Sect. 3.4).

Now we consider $J_d(n_1, n_2, n_3, n_5, n_6, n_7, n_8) = I_e(n_1, n_2, 0, n_3, n_5, n_6, n_7, n_8)$ (Fig. 12d), which is required for the calculation of the non-planar diagram $I_d$ (Fig. 11d, Sect. 3.4) without numerator. This integral vanishes when indices of the following groups of lines are non-positive: 5, 12, 23, 67, 68, 78, 26, 28, 37, 38. It becomes trivial if any of the indices is zero. Applying $\partial_3 \cdot v$, $\partial_1 \cdot v$, $\partial_2 \cdot v$, $\partial_3 \cdot k_3$, $\partial_1 \cdot k_1$, $\partial_1 \cdot (k_1 - k_2)$, $\partial_2 \cdot k_2$, $\partial_2 \cdot (k_2 - k_1)$ to the integrand, we obtain

$$\left[-2n_1\mathbf{1}^+ - 2n_2\mathbf{2}^+ + n_5\mathbf{5}^+(\mathbf{1}^- - 1)\right] J_d = 0\,, \tag{95}$$

$$\left[-2n_2\mathbf{2}^+ + n_6\mathbf{6}^+(\mathbf{2}^- - \mathbf{1}^-) + n_8\mathbf{8}^+(1 - \mathbf{1}^- + \mathbf{2}^- - \mathbf{3}^-)\right] J_d = 0\,, \tag{96}$$

$$\left[-2n_3\mathbf{3}^+ + n_7\mathbf{7}^+(\mathbf{3}^- - 1) - n_8\mathbf{8}^+(1 - \mathbf{1}^- + \mathbf{2}^- - \mathbf{3}^-)\right] J_d = 0\,, \tag{97}$$

$$\left[d - n_1 - 2n_5 + n_1\mathbf{1}^+ + n_2\mathbf{2}^+(1 - \mathbf{1}^-)\right] J_d = 0\,, \tag{98}$$

$$\left[d - n_1 - n_8 - 2n_6 + n_2\mathbf{2}^+\mathbf{1}^- + n_8\mathbf{8}^+(\mathbf{7}^- - \mathbf{6}^-)\right] J_d = 0\,, \tag{99}$$

$$\left[d - n_2 - n_6 - 2n_8 + n_2\mathbf{2}^+(\mathbf{1}^- + \mathbf{3}^- - 1) + n_6\mathbf{6}^+(\mathbf{7}^- - \mathbf{8}^-)\right] J_d = 0\,, \tag{100}$$

$$\left[d - n_3 - n_8 - 2n_7 + n_3\mathbf{3}^+ + n_8\mathbf{8}^+(\mathbf{6}^- - \mathbf{7}^-)\right] J_d = 0\,, \tag{101}$$

$$\left[d - n_3 - n_7 - 2n_8 + n_3\mathbf{3}^+(1 - \mathbf{1}^- + \mathbf{2}^-) + n_7\mathbf{7}^+(\mathbf{6}^- - \mathbf{8}^-)\right] J_d = 0\,. \tag{102}$$

Homogeneity in $\omega$ gives

$$\left[3d - n_1 - n_2 - n_3 - 2(n_5 + n_6 + n_7 + n_8) + n_1\mathbf{1}^+ + n_2\mathbf{2}^+ + n_3\mathbf{3}^+\right] J_d = 0\,. \tag{103}$$

When $n_1 < 0$ and $n_5 \neq 1$, we can raise $n_1$ by (95); when $n_1 < 0$ and $n_8 \neq 1$, we can raise $n_1$ by (97); when $n_1 < 0$ and $n_2 \neq 1$, we can raise $n_1$ by (99) (or (98)); when $n_1 < 0$ and $n_3 \neq 1$, we can raise $n_1$ by (102); when $n_1 < 0$ and $n_2 = n_3 = n_5 = n_8 = 1$, we can raise $n_1$, $n_2$ or $n_3$ by (103). When $n_2 < 0$ and $n_8 \neq 1$, we can raise $n_2$ by (97); when $n_2 < 0$ and $n_3 \neq 1$, we can raise $n_2$ by (102); when $n_2 < 0$ and $n_6 \neq 1$, we can raise $n_2$ using the sum of (96) and (97); when $n_2 < 0$ and $n_3 = n_6 = n_8 = 1$, we can raise $n_2$ or $n_8$ by (99). When $n_3 < 0$ and $n_8 \neq 1$, we can raise $n_3$ by (96); when $n_3 < 0$ and $n_7 \neq 1$, we can raise $n_3$ using the sum of (96) and (97); when $n_3 < 0$ and $n_2 \neq 1$, we can raise $n_3$ by (100); when $n_3 < 0$ and $n_2 = n_7 = n_8 = 1$, we can raise $n_3$ or $n_8$ by (101). When $n_6 < 0$ and $n_8 \neq 1$, we can raise $n_6$ by (99); when $n_6 < 0$ and $n_7 \neq 1$, we can raise $n_6$ by (102); when $n_6 < 0$ and $n_2 \neq 1$, $n_7 = n_8 = 1$, we can raise $n_6$ or $n_8$ by (96); when $n_6 < 0$ and $n_2 = n_7 = n_8 = 1$, we can raise $n_6$ or $n_2$ by (100). When $n_7 < 0$ and $n_8 \neq 1$, we can raise $n_7$ by (99); when $n_7 < 0$ and $n_6 \neq 1$, we can raise $n_7$ by (100); when $n_7 < 0$ and $n_2 \neq 1$, $n_6 = n_8 = 1$, we can raise $n_6$ or $n_8$ by (96); when

$n_7 < 0$ and $n_2 = n_6 = n_8 = 1$, we can raise $n_6$ or $n_2$ by (100). When $n_8 < 0$ and $n_6 \neq 1$, we can raise $n_8$ by (100); when $n_8 < 0$ and $n_7 \neq 1$, we can raise $n_8$ by (102); when $n_8 < 0$ and $n_2 \neq 1$, $n_6 = n_7 = 1$, we can raise $n_8$ or $n_6$ by (96); when $n_8 < 0$ and $n_2 = n_6 = n_7 = 1$, we can raise $n_6$ or $n_2$ by (100). When all the indices are positive, we can kill one of the lines 1, 6, 7 by (99).

Finally, we consider $J_e(n_1, n_2, n_3, n_5, n_6, n_7, n_8) = I_f(0, n_1, n_2, n_3, n_5, n_6, n_7, n_8)$ (Fig. 12f), which is also required for the calculation of the non-planar diagram $I_d$ (Fig. 11d, Sect. 3.4) without numerator. This integral vanishes when indices of the following groups of lines are non-positive: 5, 12, 16, 18, 67, 68, 78, 237, 328. It becomes trivial if any of the indices is zero. Applying $\partial_3 \cdot v$, $\partial_1 \cdot v$, $\partial_2 \cdot v$, $\partial_3 \cdot k_3$, $\partial_1 \cdot k_1$, $\partial_1 \cdot (k_1 - k_2)$, $\partial_2 \cdot k_2$, $\partial_2 \cdot (k_2 - k_1)$ to the integrand, we obtain

$$\left[-2n_1\mathbf{1}^+ - 2n_2\mathbf{2}^+ + n_5\mathbf{5}^+(\mathbf{2}^- - \mathbf{3}^-)\right] J_e = 0\,, \tag{104}$$

$$\left[-2n_1\mathbf{1}^+ + n_6\mathbf{6}^+(\mathbf{1}^- - \mathbf{2}^- + \mathbf{3}^- - 1) + n_8\mathbf{8}^+(\mathbf{1}^- - \mathbf{2}^-)\right] J_e = 0\,, \tag{105}$$

$$\left[-2n_2\mathbf{2}^+ - 2n_3\mathbf{3}^+ + n_7\mathbf{7}^+(\mathbf{3}^- - 1) - n_8\mathbf{8}^+(\mathbf{2}^- - \mathbf{1}^-)\right] J_e = 0\,, \tag{106}$$

$$\left[d - n_2 - 2n_5 + n_1\mathbf{1}^+(\mathbf{3}^- - \mathbf{2}^-) + n_2\mathbf{2}^+\mathbf{3}^-\right] J_e = 0\,, \tag{107}$$

$$\left[d - n_1 - n_8 - 2n_6 + n_1\mathbf{1}^+(\mathbf{2}^- - \mathbf{3}^- + 1) + n_8\mathbf{8}^+(\mathbf{7}^- - \mathbf{6}^-)\right] J_e = 0\,, \tag{108}$$

$$\left[d - n_1 - n_6 - 2n_8 + n_1\mathbf{1}^+\mathbf{2}^- + n_6\mathbf{6}^+(\mathbf{7}^- - \mathbf{8}^-)\right] J_e = 0\,, \tag{109}$$

$$\left[d - n_3 - n_8 - 2n_7 + n_2\mathbf{2}^+(1 - \mathbf{3}^-) + n_3\mathbf{3}^+ + n_8\mathbf{8}^+(\mathbf{6}^- - \mathbf{7}^-)\right] J_e = 0\,, \tag{110}$$

$$\begin{aligned}&\Big[d - n_2 - n_7 - 2n_8 + n_2\mathbf{2}^+\mathbf{1}^- + n_3\mathbf{3}^+(\mathbf{1}^- - \mathbf{2}^-)\\ &\quad + n_7\mathbf{7}^+(\mathbf{6}^- - \mathbf{8}^-)\Big] J_e = 0\,.\end{aligned} \tag{111}$$

Homogeneity in $\omega$ gives a relation identical to (103).

When $n_1 < 0$ and $n_8 \neq 1$, we can raise $n_1$ by (106); when $n_1 < 0$ and $n_6 \neq 1$, we can raise $n_1$ using the sum of (105) and (106); when $n_1 < 0$ and $n_6 = n_8 = 1$, we can raise $n_1$ or $n_6$ by (109). When $n_2 < 0$ and $n_5 \neq 1$, we can raise $n_2$ by (104); when $n_2 < 0$ and $n_8 \neq 1$, we can raise $n_2$ by (106); when $n_2 < 0$ and $n_6 \neq 1$, we can raise $n_2$ using the sum of (105) and (106); when $n_2 < 0$ and $n_1 \neq 1$, we can raise $n_2$ by (109); when $n_2 < 0$ and $n_3 \neq 1$, we can raise $n_2$ by (111); when $n_2 < 0$ and $n_1 = n_3 = n_5 = n_6 = n_8 = 1$, we can raise $n_2$, $n_1$ or $n_3$ by the homogeneity relation. When $n_3 < 0$ and $n_5 \neq 1$, we can raise $n_3$ by (104); when $n_3 < 0$ and $n_6 \neq 1$, we can raise $n_3$ by 9105); when $n_3 < 0$ and $n_7 \neq 1$, we can raise $n_3$ by (106); when $n_3 < 0$ and $n_1 \neq 1$, we can raise $n_3$ by (108); when $n_3 < 0$ and $n_2 \neq 1$, we can raise $n_3$ using the sum of (107) and (108); when $n_3 < 0$ and $n_1 = n_2 = n_5 = n_6 = n_7 = 1$, we can raise $n_3$, $n_1$ or $n_2$ by the homogeneity relation. When $n_6 < 0$ and $n_8 \neq 1$, we can raise $n_6$ by (108); when $n_6 < 0$ and $n_7 \neq 1$, we can raise $n_6$ by (111); when $n_6 < 0$ and $n_2 \neq 1$, $n_7 = n_8 = 1$, we can raise $n_6$ or $n_8$ by (105); when $n_6 < 0$ and $n_2 = n_7 = n_8 = 1$, we can raise $n_6$ or $n_2$ by (109). When $n_7 < 0$ and $n_8 \neq 1$, we can raise $n_7$ by (108); when $n_7 < 0$ and $n_6 \neq 1$, we can raise $n_7$ by (109); when $n_7 < 0$ and $n_2 \neq 1$, $n_6 = n_8 = 1$, we can raise $n_6$ or $n_8$ by (105); when $n_7 < 0$ and $n_2 = n_6 = n_8 = 1$, we can raise $n_6$ or $n_2$ by (109). When $n_8 < 0$ and $n_6 \neq 1$, we can raise $n_8$ by (109); when $n_8 < 0$ and $n_7 \neq 1$, we can raise $n_8$ by (111); when $n_8 < 0$ and $n_2 \neq 1$, $n_6 = n_7 = 1$, we can raise $n_8$ or $n_6$ by (105); when $n_8 < 0$ and $n_2 = n_6 = n_7 = 1$, we can raise $n_6$ or $n_2$ by (109). When all the indices are positive, we can kill one of the lines 2, 7, 8 by (108).

### 3.6 Diagrams with five heavy-quark lines

Let's define (Fig. 11h)

$$\begin{aligned}
&\int \frac{N_{12}^{n_{12}} N_{13}^{n_{13}} N_{23}^{n_{23}} d^d k_1 d^d k_2 d^d k_3}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7} D_8^{n_8}}\\
&= -i\pi^{3d/2} (-2\omega)^{3d+2(n_{12}+n_{13}+n_{23})-2\sum_{i=6}^{8} n_i}\\
&\times I_h(n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8; n_{12}, n_{13}, n_{23})\,,\\
&D_1 = (k_2+p)\cdot v/\omega\,, \quad D_2 = (k_1+k_2+p)\cdot v/\omega\,, \quad D_3 = (k_1+p)\cdot v/\omega\,,\\
&D_4 = (k_1+k_3+p)\cdot v/\omega\,, \quad D_5 = (k_3+p)\cdot v/\omega\,,\\
&D_6 = -k_2^2\,, \quad D_7 = -k_3^2\,, \quad D_8 = -k_1^2\,,\\
&N_{12} = 2k_1\cdot k_2\,, \quad N_{13} = 2k_1\cdot k_3\,, \quad N_{23} = 2k_2\cdot k_3\,.
\end{aligned} \tag{112}$$

This integral is mirror-symmetric with respect to $1 \leftrightarrow 5$, $2 \leftrightarrow 4$, $6 \leftrightarrow 7$. It vanishes when the indices of the following groups of lines are non-positive: 6, 7, 8, 12, 45, 234. There are two linear relations among the heavy-quark denominators:

$$\left[1 - \mathbf{1}^- + \mathbf{2}^- - \mathbf{3}^-\right] I_h = 0\,, \quad \left[1 - \mathbf{3}^- + \mathbf{4}^- - \mathbf{5}^-\right] I_h = 0\,. \tag{113}$$

If $n_1 < 0$, $n_2 < 0$, $n_3 < 0$, $n_4 < 0$ or $n_5 < 0$, it can be raised by (113). Otherwise, one of the lines 1, 2, 3 (or 3, 4, 5) can be killed by (113). In the cases $n_1 = 0$ or $n_2 = 0$, we can use (113) to kill one of the lines 3, 4, 5; in the case $n_3 = 0$ (Fig. 12f), we can use

$$\left[\mathbf{1}^- - \mathbf{2}^- + \mathbf{3}^- - \mathbf{4}^-\right] I_h = 0 \tag{114}$$

to raise, say, $n_4$ and kill one of the lines 1, 2, 3. We are left with the integrals (Fig. 13)

$$I_h(0, n_2, 0, n_4, n_5, n_6, n_7, n_8; n_{12}, n_{13}, n_{23}) =$$

$$K_c(n_2,n_5,n_4,n_6,n_7,n_8;n_{23},n_{12},n_{13})\,,$$
$$I_h(0,n_2,n_3,0,n_5,n_6,n_7,n_8;n_{12},n_{13},n_{23}) =$$
$$K_b(n_5,n_3,n_2,n_7,n_8,n_6;n_{13},n_{23},n_{12})\,,$$
$$I_h(0,n_2,n_3,n_4,0,n_6,n_7,n_8;n_{12},n_{13},n_{23}) =$$
$$K_c(n_2,n_4,n_3,n_6,n_7,n_8;n_{23},n_{12},n_{13})\,,$$
$$I_h(n_1,0,0,n_4,n_5,n_6,n_7,n_8;n_{12},n_{13},n_{23}) =$$
$$K_b(n_1,n_5,n_4,n_6,n_7,n_8;n_{23},n_{12},n_{13})\,,$$
$$I_h(n_1,0,n_3,0,n_5,n_6,n_7,n_8;n_{12},n_{13},n_{23}) =$$
$$K_a(n_1,n_5,n_3,n_6,n_7,n_8;n_{23},n_{12},n_{13})\,,$$

which will be discussed later in this Section (the case $n_2 = n_5 = 0$ is symmetric to $n_1 = n_4 = 0$, and $n_3 = n_4 = 0$ — to $n_2 = n_3 = 0$).

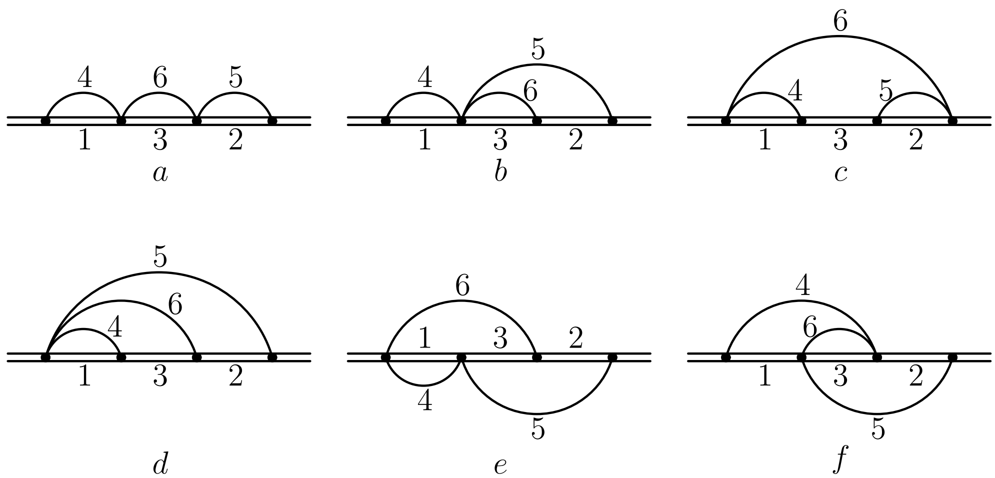



Figure 13: Reduced forms of the diagrams Fig. 11h–j

Let's define (Fig. 11i)

$$\int \frac{N_{12}^{n_{12}} N_{13}^{n_{13}} N_{23}^{n_{23}} d^d k_1 d^d k_2 d^d k_3}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7} D_8^{n_8}}$$
$$= -i\pi^{3d/2} (-2\omega)^{3d+2(n_{12}+n_{13}+n_{23})-2\sum_{i=6}^{8} n_i}$$
$$\times\, I_i(n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8;n_{12},n_{13},n_{23})\,,$$
$$D_1 = (k_1+p)\cdot v/\omega\,,\quad D_2 = (k_1+k_2+p)\cdot v/\omega\,,$$
$$D_3 = (k_1+k_2+k_3+p)\cdot v/\omega\,,\quad D_4 = (k_1+k_3+p)\cdot v/\omega\,,$$
$$D_5 = (k_2+p)\cdot v/\omega\,,\quad D_6 = -k_1^2\,,\quad D_7 = -k_2^2\,,\quad D_8 = -k_3^2\,. \quad (115)$$

This integral vanishes when the indices of the following groups of lines are non-positive: 6, 7, 8, 34, 123. There are two linear relations among the heavy-quark denominators:

$$\left[1-\mathbf{1}^-+\mathbf{3}^--\mathbf{4}^-\right]I_i=0\,,\quad \left[1-\mathbf{1}^-+\mathbf{2}^--\mathbf{5}^-\right]I_i=0\,. \tag{116}$$

If $n_1<0$, $n_2<0$, $n_3<0$, $n_4<0$, or $n_5<0$, it can be raised by (116). Otherwise, one of the lines 1, 3, 4 (or 1, 2, 5) can be killed by (116). In the cases $n_3=0$ or $n_4=0$, we can use (116) to kill one of the lines 1, 2, 5; in the case $n_1=0$, we can use

$$\left[\mathbf{2}^--\mathbf{3}^-+\mathbf{4}^--\mathbf{5}^-\right]I_i=0 \tag{117}$$

to raise, say, $n_5$ and kill one of the lines 2, 3, 4. We are left with the integrals (Fig. 13)

$$\begin{aligned}
&I_i(0,0,n_3,n_4,n_5,n_6,n_7,n_8;n_{12},n_{13},n_{23})=\\
&\quad K_d(n_3,n_5,n_4,n_6,n_7,n_8;n_{12},n_{13},n_{23})\,,\\
&I_i(0,n_2,0,n_4,n_5,n_6,n_7,n_8;n_{12},n_{13},n_{23})=\\
&\quad K_c(n_2,n_4,n_5,n_6,n_8,n_7;n_{13},n_{12},n_{23})\,,\\
&I_i(0,n_2,n_3,0,n_5,n_6,n_7,n_8;n_{12},n_{13},n_{23})=\\
&\quad K_d(n_3,n_5,n_2,n_8,n_7,n_6;n_{23},n_{13},n_{12})\,,\\
&I_i(n_1,0,0,n_4,n_5,n_6,n_7,n_8;n_{12},n_{13},n_{23})=\\
&\quad K_b(n_1,n_5,n_4,n_6,n_7,n_8;n_{12},n_{13},n_{23})\,,\\
&I_i(n_1,0,n_3,0,n_5,n_6,n_7,n_8;n_{12},n_{13},n_{23})=\\
&\quad K_f(n_1,n_5,n_3,n_6,n_7,n_8;n_{12},n_{13},n_{23})\,,\\
&I_i(n_1,n_2,0,n_4,0,n_6,n_7,n_8;n_{12},n_{13},n_{23})=\\
&\quad K_e(n_4,n_1,n_2,n_8,n_6,n_7;n_{13},n_{23},n_{12})\,,\\
&I_i(n_1,n_2,n_3,0,0,n_6,n_7,n_8;n_{12},n_{13},n_{23})=\\
&\quad K_d(n_3,n_1,n_2,n_8,n_6,n_7;n_{13},n_{23},n_{12})\,.
\end{aligned}$$

The integrals $K$ are shown in Fig. 13; the momenta of lines 4, 5, 6 are $k_1$, $k_2$, $k_3$. They are equal to

$$\begin{aligned}
&K_a(n_1,n_2,n_3,n_4,n_5,n_6;n_{12},n_{13},n_{23})=(\mathbf{0}^+)^{n_{12}}(\mathbf{6}^--\mathbf{4}^--\mathbf{8}^-)^{n_{13}}\\
&\quad\times(\mathbf{7}^--\mathbf{5}^--\mathbf{8}^-)^{n_{23}}I_c(n_1,n_2,n_3,n_4,n_5,0,0,n_6)\,,\\
&K_b(n_1,n_2,n_3,n_4,n_5,n_6;n_{12},n_{13},n_{23})=(\mathbf{0}^+)^{n_{12}}(\mathbf{6}^--\mathbf{4}^--\mathbf{8}^--\mathbf{0}^+)^{n_{13}}\\
&\quad\times(\mathbf{7}^--\mathbf{5}^--\mathbf{8}^-)^{n_{23}}I_c(n_1,n_2,n_3,n_4,n_5,0,n_6,0)\,,\\
&K_c(n_1,n_2,n_3,n_4,n_5,n_6;n_{12},n_{13},n_{23})=(\mathbf{4}^-+\mathbf{5}^--\mathbf{6}^--\mathbf{7}^-+\mathbf{0}^+)^{n_{12}}\\
&\quad\times(\mathbf{6}^--\mathbf{4}^-+\mathbf{8}^-)^{n_{13}}(\mathbf{7}^--\mathbf{5}^-+\mathbf{8}^-)^{n_{23}}I_c(n_1,n_2,n_3,0,0,n_4,n_5,n_6)\,,\\
&K_d(n_1,n_2,n_3,n_4,n_5,n_6;n_{12},n_{13},n_{23})=(\mathbf{6}^-+\mathbf{7}^--\mathbf{4}^--\mathbf{5}^--\mathbf{0}^+)^{n_{13}}
\end{aligned}$$

$$\times (\mathbf{5}^- - \mathbf{7}^- + \mathbf{8}^- + \mathbf{0}^+)^{n_{12}} (\mathbf{5}^- + \mathbf{7}^- - \mathbf{8}^-)^{n_{23}} I_c(n_1, n_2, n_3, n_4, n_5, 0, 0, n_6)\,,$$

$$K_e(n_1, n_2, n_3, n_4, n_5, n_6; n_{12}, n_{13}, n_{23}) = (\mathbf{7}^- - \mathbf{5}^- + \mathbf{8}^-)^{n_{12}} (\mathbf{6}^- - \mathbf{4}^- + \mathbf{8}^-)^{n_{13}}$$

$$\times (\mathbf{5}^- - \mathbf{7}^- + \mathbf{8}^- + \mathbf{0}^+)^{n_{23}} I_d(n_1, n_2, n_3, 0, n_5, n_6, 0, n_4)\,,$$

$$K_f(n_1, n_2, n_3, n_4, n_5, n_6; n_{12}, n_{13}, n_{23}) = (\mathbf{0}^+)^{n_{12}} (\mathbf{6}^- - \mathbf{4}^- - \mathbf{8}^-)^{n_{13}}$$

$$\times (\mathbf{7}^- - \mathbf{5}^- - \mathbf{8}^-)^{n_{23}} I_d(n_1, n_2, n_3, n_4, n_5, 0, 0, n_6)\,.$$

We could also express $K_b$ and $K_d$ via $I_b$ or $I_d$, and $K_c$ via $I_b$.

Finally, let's define (Fig. 11j)

$$\begin{aligned}
&\int \frac{N_{12}^{n_{12}} N_{13}^{n_{13}} N_{23}^{n_{23}} d^d k_1 d^d k_2 d^d k_3}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7} D_8^{n_8}} \\
&= -i\pi^{3d/2} (-2\omega)^{3d+2(n_{12}+n_{13}+n_{23})-2\sum_{i=6}^{8} n_i} \\
&\times I_j(n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8; n_{12}, n_{13}, n_{23})\,, \\
&D_1 = (k_1 + p)\cdot v/\omega\,, \quad D_2 = (k_1 + k_2 + p)\cdot v/\omega\,, \\
&D_3 = (k_1 + k_2 + k_3 + p)\cdot v/\omega\,, \quad D_4 = (k_2 + k_3 + p)\cdot v/\omega\,, \\
&D_5 = (k_3 + p)\cdot v/\omega\,, \quad D_6 = -k_1^2\,, \quad D_7 = -k_2^2\,, \quad D_8 = -k_3^2\,.
\end{aligned} \tag{118}$$

This integral is mirror-symmetric with respect to $1 \leftrightarrow 5$, $2 \leftrightarrow 4$, $6 \leftrightarrow 8$. It vanishes when the indices of the following groups of lines are non-positive: 6, 7, 8, 123, 234, 345. The cases

$$\begin{aligned}
I_j(n_1, n_2, n_3, n_4, 0, n_6, n_7, n_8) &= I_i(n_1, n_2, n_3, n_4, 0, n_6, n_7, n_8)\,, \\
I_j(n_1, 0, n_3, n_4, n_5, n_6, n_7, n_8) &= I_i(n_1, 0, n_3, n_4, n_5, n_6, n_8, n_7)\,, \\
I_j(n_1, n_2, 0, n_4, n_5, n_6, n_7, n_8) &= I_h(n_1, n_2, 0, n_4, n_5, n_6, n_8, n_7)\,,
\end{aligned}$$

as well as the cases $n_1 = 0$, $n_4 = 0$, symmetric to the first two ones, reduce to the previously considered integrals. There are two linear relations among the heavy-quark denominators:

$$\left[1 - \mathbf{1}^- + \mathbf{3}^- - \mathbf{4}^-\right] I_j = 0\,, \quad \left[1 - \mathbf{2}^- + \mathbf{3}^- - \mathbf{5}^-\right] I_j = 0\,. \tag{119}$$

If $n_1 < 0$, $n_2 < 0$, $n_3 < 0$, $n_4 < 0$, or $n_5 < 0$, it can be raised by (119). Otherwise, one of the lines 1, 3, 4 (or 2, 3, 5) can be killed by (119).

# 4 Implementation and testing

The package Grinder consists of a set of mutually recursive procedures for Feynman integrals of various topologies, which reduce a given Feynman integral to simpler ones, until boundary-case integrals with known values are reached. Remembering results of previous function calls may make the program run much faster, if there is enough memory. The package is written in REDUCE [22, 23]; implementation of remember-tables required some symbolic-mode programming.

I also re-implemented it in Axiom [24]. All expressions involved are linear combinations of basis integrals with coefficients which are rational functions of $d$. It is convenient to use Axiom `Vector` domain, which has all the necessary operations. This makes intermediate expressions shorter than in the case when multivariate rational functions are used for entire expressions, because, typically, not all basis integrals are accompanied by every possible denominator. The amount of GCD calculations is thus reduced. This improvement can be, in principle, back-propagated to the REDUCE implementation by using matrices. However, working with matrices in REDUCE is awkward, because there are no local matrix variables, and no easy way for a function to return a matrix. On the other hand, complete diagram calculations, including tensor and $\gamma$-matrix algebra, can be done in REDUCE.

The main method of testing was checking various recurrence relations (including those which were not directly used for construction of the algorithm) in nested loops over $n_i$. For each of two-loop and generalized two-loop integrals, which depend on 5 indices, a typical number of checks was about 20000; each test set of this size runs for a few hours. Integrals with 8 or 9 indices are more difficult to check. Some test sets required a few days of CPU time. A tool showing how many times each linear code segment has been executed would be invaluable for setting up test cases which check all branches at least once. Unfortunately, such a tool is not available in either programming system, and I had to emulate it by hand.

I am grateful to P. A. Baikov, D. J. Broadhurst, K. G. Chetyrkin, S. A. Larin and J. A. M. Vermaseren for useful discussions, and to INTAS for the grant which allowed me to buy Axiom.

# References


[1] R. Harlander and M. Steinhauser, *Prog. Part. Nucl. Phys.* **43** (1999) 167, hep-ph/9812357.

[2] G. 't Hooft, M. Veltman, *Nucl. Phys.* **B44** (1972) 189.

[3] F. V. Tkachov, *Phys. Lett.* **B100** (1981) 65; K. G. Chetyrkin and F. V. Tkachov, *Nucl. Phys.* **B192** (1981) 159.

[4] M. Veltman, SCHOONSCHIP (CERN, 1967); H. Strubbe, *Comp. Phys. Comm.* **8** (1974) 1.

[5] S. G. Gorishny, S. A. Larin and F. V. Tkachov, Preprint INR P-0330 (Moscow, 1984); S. G. Gorishny, S. A. Larin, L. R. Surguladze and F. V. Tkachov, *Comp. Phys. Comm.* **55** (1989) 381.

[6] S. A. Larin, F. V. Tkachov and J. A. M. Vermaseren, Preprint NIKHEF-H/91-18 (Amsterdam, 1991).

[7] J. A. M. Vermaseren, Symbolic Manipulation with FORM (Amsterdam, 1991).

[8] D. J. Broadhurst, *Zeit. Phys.* **C54** (1992) 599.

[9] J. Fleischer and O. V. Tarasov, *Comp. Phys. Comm.* **71** (1992) 193.

[10] K. G. Chetyrkin, J. H. Kühn and M. Steinhauser, *Nucl. Phys.* **B482** (1996) 213.

[11] S. Laporta and E. Remiddi, *Phys. Lett.* **B379** (1996) 283.

[12] K. Melnikov, T. van Ritbergen, Preprint SLAC-PUB-8281, TTP99-44, hep-ph/9911277; SLAC-PUB-8321, TTP99-51, hep-ph/9912391.

[13] M. Neubert, *Phys. Reports* **245** (1994) 259; A. V. Manohar and M. B. Wise, Heavy Quark Physics (Cambridge University Press, Cambridge, 2000).

[14] A. G. Grozin, Introduction to Heavy Quark Effective Theory, Preprint Budker INP 92-97 (Novosibirsk, 1992), hep-ph/9908366; Recent progress on HQET Lagrangian, *Proc. IV Int. Workshop on Progress in Heavy Quark Physics*, ed. M. Beyer, T. Mannel and H. Schröder (Univ. of Rostock, 1998), p. 93, hep-ph/9711269.

[15] D. J. Broadhurst and A. G. Grozin, *Phys. Lett.* **B267** (1991) 105.

[16] D. J. Broadhurst and A. G. Grozin, Multiloop Calculations in Heavy Quark Effective Theory, in *New Computing Technique in Physics Research IV*, ed. B. Denby and D. Perret-Gallix (World Scientific, Singapore, 1995), p. 217, hep-ph/9504400.

[17] S. A. Larin, private communication.

[18] M. Beneke and V. M. Braun, *Nucl. Phys.* **B426** (1994) 301.

[19] K. G. Chetyrkin, A. L. Kataev and F. V. Tkachov, *Nucl. Phys.* **B174** (1980) 345.

[20] D. J. Broadhurst, hep-th/9604128; hep-th/9612012; J. M. Borwein, D. M. Bradley and D. J. Broadhurst, *Electronic J. Combinatorics* **4** (1997) R5, hep-th/9611004.

[21] P. A. Baikov, Preprint INP MSU-99-45/603, TTP99-52, hep-ph/9912421.

[22] A. C. Hearn, REDUCE User's Manual, Version 3.7 (1999).

[23] A. G. Grozin, Using REDUCE in High Energy Physics (Cambridge University Press, Cambridge, 1997).

[24] R. D. Jenks and R. S. Sutor, Axiom: The Scientific Computation System (Springer-Verlag, 1992).